

# Matroids of Gain Graphs in Applied Discrete Geometry

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## Abstract

A  $\Gamma$ -gain graph is a graph whose oriented edges are labeled invertibly from a group  $\Gamma$ . Zaslavsky proposed two matroids of  $\Gamma$ -gain graphs, called gain matroids and lift matroids, and investigated linear representations of them. Each matroid has a canonical representation over a field  $\mathbb{F}$  if  $\Gamma$  is isomorphic to a subgroup of  $\mathbb{F}^\times$  in the case of gain matroids or  $\Gamma$  is isomorphic to an additive subgroup of  $\mathbb{F}$  in the case of lift matroids. The canonical representation of the gain matroid of a complete graph is also known as a Dowling geometry, as it was first introduced by Dowling for finite groups  $\Gamma$ .

In this paper, we extend these matroids in two ways. The first one is extending the rank function of each matroid, based on submodular functions over  $\Gamma$ . The resulting rank function generalizes that of the union of gain matroids or lift matroids. Another one is extending the canonical linear representation of the union of  $d$  copies of a gain matroid or a lift matroid, based on linear representations of  $\Gamma$  on a  $d$ -dimensional vector space. We show that linear matroids of the latter extension are indeed special cases of the first extensions, as in the relation between Dowling geometries and gain matroids. We also discuss an attempt to unify the extension of gain matroids and that of lift matroids.

This work is motivated from recent research on the combinatorial rigidity of symmetric graphs. As special cases, we give several new results on this topic, including combinatorial characterizations of the symmetric-rigidity of generic body-bar frameworks with point group symmetries or crystallographic symmetries and the symmetric parallel redrawability of generic bar-joint frameworks with point group symmetries or crystallographic symmetries.

## 1 Introduction

A  $\Gamma$ -gain graph  $(G, \psi)$  is a pair of a graph  $G = (V, E)$  and an assignment  $\psi$  of an element of a group  $\Gamma$  with each oriented edge such that reversing the direction inverts the gain. Gain graphs are also known as group-labeled graphs and appear in wide range of combinatorial problems and applications. Zaslavsky [42, 43] studied a class of matroids of graphs, called *biased matroids*, and as a principal case he introduced a matroid of a  $\Gamma$ -gain graph  $(G, \psi)$ , called the *gain matroid*  $\mathbf{G}(G, \psi)$  of  $(G, \psi)$ . Gain matroids include several known matroids on group-labeled graphs, such as bicircular matroids, Dowling geometries, and matroids on signed graphs. Zaslavsky [43] also proposed another matroid on a  $\Gamma$ -gain graph  $(G, \psi)$ , called the lift matroid  $\mathbf{L}(G, \psi)$ , which can be naturally constructed from the graphic matroid of  $G$  by an elementary lift.

Each matroid has a canonical representation over a field  $\mathbb{F}$  if  $\Gamma$  is isomorphic to a subgroup of the multiplicative group  $\mathbb{F}^\times$  of  $\mathbb{F}$  in the case of  $\mathbf{G}(G, \psi)$  or  $\Gamma$  is isomorphic to

an additive subgroup of  $\mathbb{F}$  in the case of  $\mathbf{L}(G, \psi)$ . The canonical representations of the gain matroid of a dense graph is also known as a Dowling geometry, as it was first introduced by Dowling [7] for finite groups  $\Gamma$ .

As a further extension, Whittle [41] discussed a counterpart of biased matroids in general matroids, by extending the construction of biased matroids from graphic matroids.

In this paper, we shall consider extensions, sticking to gain graphs. We propose matroids of gain graphs, extending the constructions of gain matroids or lift matroids in the following two ways. The first one is extending the rank function of each matroid, based on submodular functions over  $\Gamma$ . The resulting rank function generalizes that of the union of gain matroids or lift matroids. Another one is extending the canonical linear representation of the union of  $d$  copies of a gain matroid or a lift matroid, based on linear representations of  $\Gamma$  on a  $d$ -dimensional vector space. We show that linear matroids of the latter extension are indeed special cases of the first extensions, as in the relation between Dowling geometries and gain matroids.

This work is motivated from recent research on the combinatorial rigidity of symmetric graphs. Characterizing generic rigidity of graphs is one of central problems in rigidity theory, where a graph is identified with a *bar-joint framework* by regarding each vertex as a joint and each edge as a bar. In this context, a bar-joint framework is denoted by a pair  $(G, p)$  of a graph  $G = (V, E)$  and  $p : V \rightarrow \mathbb{R}^d$  that specifies the position of each vertex. For 2-dimensional rigidity, Laman's theorem asserts that  $(G, p)$  is minimally infinitesimally rigid on any generic  $p : V \rightarrow \mathbb{R}^2$  if and only if  $|E| = 2|V| - 3$  and  $|F| \leq 2|V(F)| - 3$  for any  $F \subseteq E$ , where  $V(F)$  denotes the set of vertices incident to edges in  $F$ . However, in spite of exhausting efforts so far, the 3-dimensional counterpart has not been obtained yet.

Although characterizing generic 3-dimensional rigidity of graphs is recognized as one of the most difficult open problems in this field, there are solvable structural models even in higher dimension. The most important case is a *body-bar framework* proposed by Tay [36]. A body-bar framework is a structural model consisting of disjoint rigid bodies articulated by bars, and the underlying graph is extracted by associating each body with a vertex and each bar with an edge. Tay [36] proved that a generic body-bar framework (i.e., relative positions of bars are generic) is rigid if the underlying graph has rank  $\binom{d+1}{2}(|V| - 1)$  in the union of  $\binom{d+1}{2}$  copies of the graphic matroid.

There are several attempts to extend these results to *symmetric frameworks*. Symmetric frameworks are those which are invariant with an action of a point group in finite case or of a space group in infinite case. For a finite case, initiated by a combinatorial necessary condition [10, 6], Schulze [30, 31] shows an extension of Laman's theorem of minimal 2-dimensional rigidity subject to certain point group symmetries.

Recently, motivated from the study of infinite frameworks, the concept of "symmetry-preserving" infinitesimal rigidity (simply called *symmetric infinitesimal rigidity*) has been proposed for finite frameworks [33] and for infinite periodic frameworks [2], where each infinitesimal motion is also subject to the underlying symmetry. It was proved that the symmetric infinitesimal rigidity of symmetric frameworks can be checked by computing the rank of linear matroids defined on the edge sets of the underlying quotient gain graphs, and thus can be analyzed as in a conventional manner. For infinite periodic frameworks, there are other attempts to combinatorially capture the rigidity, see e.g., [24].

After the concept of symmetric infinitesimal rigidity has been emerged, characterizing *symmetric generic rigidity* (i.e., symmetric infinitesimal rigidity on generic symmetric con-

figurations) in terms of the underlying quotient gain graphs were proved by Ross [26, 27] for periodic 2-dimensional bar-joint frameworks and periodic 3-dimensional body-bar frameworks with fixed lattice metric and by Malestein and Theran [20, 21] for crystallographic 2-dimensional bar-joint frameworks with flexible lattice metric.

The result of this paper is indeed inspired by these previous results. As shown by Tay [36] or Whiteley [37, 39], the union of copies of graphic matroids plays a central role in combinatorial rigidity theory, that is, most known combinatorial characterizations are written in terms of the union of copies of graphic matroids or its variants, called *count matroids* (see e.g., [11] for count matroids). It is thus natural to investigate the union of copies of gain matroids to derive the symmetric analogues on gain graphs. However, when compared with the canonical linear representation of the union of gain matroids (see §3), linear matroids of gain graphs proposed in [33, 27, 21, 4] much rely on algebraic structures of the underlying groups in their definitions. The primary motivation of this paper is to propose a class of gain matroids, which forms the foundation in the study of symmetric generic rigidity, as the union of graphic matroids does in conventional rigidity problem.

As another application, we shall also consider the symmetric version of the *parallel redrawing problem*. In the parallel redrawing problem, we are concerned about whether a given straight-line drawing of a graph in the Euclidean space admits a parallel redrawing, that is, another straight-line drawing such that each edge is parallel to the corresponding one in the original drawing. Since any drawing admits a parallel redrawing by a translation or a dilation, we are asked whether all possible parallel redrawing are obtained in these trivial ways. In the context of rigidity theory, the concept of parallel redrawability is known as the *direction-rigidity* of bar-joint frameworks, where we are interested in direction-constraint, rather than conventional length-constraint, and Whiteley [39] proved a combinatorial characterization for generic frameworks. Here, we shall discuss the symmetric counterpart, called the *symmetric parallel redrawing problem*, where both drawing and its redrawing are subject to a symmetry. Malestein and Theran [21] studied it in  $d = 2$ .

We list applications addressed in this papers: the  $d$ -dimensional symmetric parallel redrawing problem with point group symmetry (§6.3); the 2-dimensional symmetric generic rigidity of bar-joint frameworks with rotational symmetry (§6.4); the  $d$ -dimensional symmetric generic rigidity of body-bar frameworks with crystallographic symmetry and fixed lattice metric (§7.2); the  $d$ -dimensional symmetric parallel redrawing problem with symmetric crystallographic symmetry and flexible lattice metric (§10.2); the 2-dimensional symmetric generic rigidity of bar-joints frameworks with crystallographic symmetry whose linear part is a group of rotations (§10.2). The results contain existing works on symmetric rigidity, except [16] and [3, 4]: in [16] combinatorial characterizations of the 2-dimensional symmetric rigidity of bar-joint frameworks with point group symmetry are discussed, and in [3, 4] combinatorial characterizations of the  $d$ -dimensional rigidity of periodic frameworks are discussed in terms of quotient graphs (but not quotient gain graphs). We believe that matroids of gain graphs proposed in this paper can be used to derive other symmetric analogues of combinatorial characterizations appeared in combinatorial geometry (see e.g. [39]).

The paper is organized as follows. In §2 and §3, we briefly review fundamental facts on gain graphs and (poly)matroids, respectively. In particular, we shall explain details of matroids induced by monotone submodular functions in §3, as our extensions belong to this class.

Extensions of gain matroids and lift matroids are described in §4, §5, §8 and §9, and the remaining sections are devoted to applications. In §4, we give an extension of rank functions of gain matroids via submodular functions over groups, while in §5 we give an extension of Dowling geometries via group representations. We give a combinatorial characterization (Theorem 5.4) of the proposed linear matroids, which implies that these are special cases of matroids in §4. Proving such a characterization does not look an easy task at a glance, but it turns out, by using the polymatroid theory discussed in §3, that the problem is as easy as the case of gain matroids. As applications, we will discuss the parallel redrawing problem and the symmetric rigidity problem of bar-joint frameworks with point group symmetry in §6. In §7, we also discuss an application to the symmetric rigidity problem of body-bar frameworks with crystallographic symmetry.

§8 gives counterparts of these results for lift matroids. In §9, we attempt to unify the extension of gain matroids and that of lift matroids, based on the representation theory obtained so far. In §10, we give further applications to the parallel redrawing problem or the rigidity problem of bar-joint frameworks with crystallographic symmetry.

We conclude introduction by listing some notations used throughout the paper. A *partition*  $\mathcal{P}$  of a finite set  $E$  is a set of nonempty subsets of  $E$  such that each element of  $E$  belongs to exactly one subset of  $\mathcal{P}$ . If  $E = \emptyset$ , the partition of  $E$  is defined as the empty set. A *subpartition* of  $E$  is a partition of a subset of  $E$ .

For an undirected graph  $G$ ,  $V(G)$  and  $E(G)$  denotes the vertex set and the edge set of  $G$ , respectively. For  $F \subseteq E(G)$ ,  $V(F)$  denotes the set of endvertices of edges in  $F$ , and let  $G[F] = (V(F), F)$ , that is, the graph edge-induced by  $F$ .

For simplicity of description, we shall use some terminologies for referring edge subsets, which are conventionally used for subgraphs, as follows. Let  $F \subseteq E$ .  $F$  is called *connected* if  $G[F]$  is connected. A *connected component* of  $F$  is the edge set of a connected component of  $G[F]$ .  $C(F)$  denotes the partition of  $F$  into connected components of  $F$ , and let  $c(F) = |C(F)|$ .  $F$  is called a *forest* if it contains no cycle and called a *tree* if it is connected and forest.  $F$  is called a *spanning tree* of a graph  $G = (V, E)$  if  $F$  is a tree with  $F \subseteq E$  and  $V(F) = V$ .

A graph is called *simple* if it contains neither a loop nor parallel edges. In a simple undirected graph, an edge between  $i$  and  $j$  is denoted by  $\{i, j\}$ . Similarly, in a simple directed graph, an edge oriented from  $i$  to  $j$  is denoted by  $(i, j)$ . Even though the graph is not simple, we sometimes denote  $e = (i, j)$  to mean that an edge  $e$  is oriented from  $i$  to  $j$ , if it is clear from the context.

Throughout the paper, let  $\mathbb{K}$  be a field, which may be finite, and  $\mathbb{F}$  a subfield of  $\mathbb{K}$  such that  $\mathbb{K}$  has transcendentals  $\alpha_1, \dots, \alpha_k$  that form an algebraically independent set over  $\mathbb{F}$ , where  $k$  is finite and becomes clear from the context (i.e., it depends on the size of ground sets).

For a group  $\Gamma$  and  $X \subseteq \Gamma$ ,  $\langle X \rangle$  denotes a subgroup of  $\Gamma$  generated by  $X$ .

## 2 Fundamentals on Gain Graphs

In this section we shall review properties of gain graphs. See e.g., [13, 42, 43] for concrete explanations on this topic. The following description is based on [16].

## 2.1 Gain graphs

Let  $G = (V, E)$  be a directed graph which may contain multiple edges and loops, and let  $\Gamma$  be a group. A pair is called a  $\Gamma$ -*gain graph*  $(G, \psi)$ , in which each edge is associated with an element of  $\Gamma$  by a *gain function*  $\psi : E \rightarrow \Gamma$ . See Figure 1 for an example. A gain graph is a directed graph, but its orientation is used only for the reference of the gain labeling. Namely, we can change orientation of each edge as we like, by imposing a property to  $\psi$  such that, if an edge in one direction has label  $g$ , then it has  $g^{-1}$  in the other direction. Thus, we often do not distinguish  $G$  and the underlying undirected graph and use notations in the introduction, which were introduced for undirected graphs, if it is clear from the context.

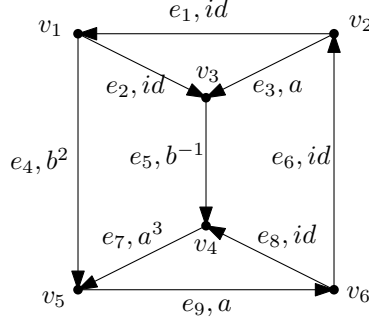


Figure 1: An example of an  $\Gamma$ -gain graph, where  $\Gamma$  is a group generated by  $a$  and  $b$ .

A *walk* is a sequence  $W = v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k$  of vertices and edges such that  $v_i$  is incident to  $e_i$  and  $e_{i+1}$  for every  $1 \leq i \leq k$  and  $v_0$  and  $v_k$  are incident to  $e_1$  and  $e_k$ , respectively. The reversed walk of  $W$  is  $W^{-1} = v_k, e_k, \dots, e_1, v_0$ . For two walks  $W$  and  $W'$  for which the end vertex of  $W$  and the starting vertex of  $W'$  coincide, we denote the concatenation of  $W$  and  $W'$  (that is, the walk  $W$  followed by  $W'$ ) by  $W * W'$ . A walk is called *closed* if the starting vertex and the end vertex coincide. The *gain* of a walk  $W$  is defined as  $\psi(W) = \psi(e_1) \cdot \psi(e_2) \cdots \psi(e_k)$  if each edge is oriented in the forward direction through  $W$ , and for a backward edge  $e_i$  we replace  $\psi(e_i)$  with  $\psi(e_i)^{-1}$  in the formulation. For example, in Figure 1,  $W = e_2, e_5, e_7, e_4$  is a closed walk starting at  $v_1$  and its gain is  $b^{-1}a^3b^{-2}$ .

Let  $(G, \psi)$  be a gain graph, and let  $v \in V(G)$ . We denote by  $\pi(G, v)$  the set of closed walks starting at  $v$ . Similarly, for  $X \subseteq E(G)$  and  $v \in V(G)$ ,  $\pi(X, v)$  denotes the set of closed walks starting at  $v$  and using only edges of  $X$ , where  $\pi(X, v) = \emptyset$  if  $v \notin V(X)$ .

Let  $X \subseteq E(G)$ . The *gain set* of  $X$  relative to  $v$  is defined as  $\psi(X, v) = \{\psi(W) \mid W \in \pi(X, v)\}$ , and  $\langle X \rangle_{\psi, v}$  is defined as the subgroup of  $\Gamma$  generated by the elements of  $\psi(X, v)$ .

**Proposition 2.1.** *For any connected  $X \subseteq E(G)$  and two vertices  $u, v \in V(X)$ ,  $\langle X \rangle_{\psi, u}$  is conjugate to  $\langle X \rangle_{\psi, v}$ .*

## 2.2 Switching operations

For  $v \in V(G)$  and  $g \in \Gamma$ , a *switching* at  $v$  with  $g$  changes the gain function  $\psi$  on  $E(G)$  as follows:

$$\psi'(e) = \begin{cases} g \cdot \psi(e) & \text{if } e \text{ is directed from } v \\ \psi(e) \cdot g^{-1} & \text{if } e \text{ is directed to } v \\ \psi(e) & \text{otherwise.} \end{cases} \quad (1)$$

By definition,  $\psi'(e) = g \cdot \psi(e) \cdot g^{-1}$  if  $e$  is a loop attached at  $v$ . We say that a gain function  $\psi'$  on  $E(G)$  is *equivalent* to another gain function  $\psi$  on  $E(G)$  if  $\psi'$  is obtained from  $\psi$  by a sequence of switchings.

**Proposition 2.2.** *Let  $(G, \psi)$  be a gain graph. Let  $\psi'$  be a gain function equivalent to  $\psi$ . Then, for any  $X \subseteq E(G)$  and any  $v \in V(G)$ ,  $\langle X \rangle_{\psi', v}$  is conjugate to  $\langle X \rangle_{\psi, v}$ .*

**Proposition 2.3.** *Let  $(G, \psi)$  be a gain graph. Then, for any forest  $F \subseteq E(G)$ , there is an equivalent gain function  $\psi'$  on  $E(G)$  such that  $\psi'(e) = \text{id}$  for every  $e \in F$ .*

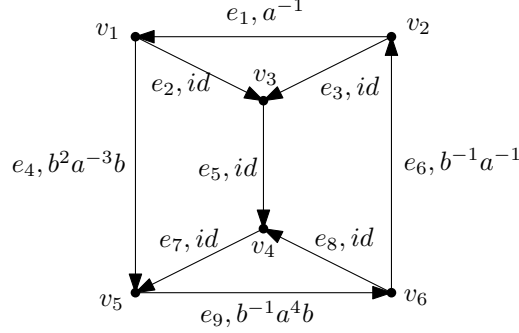


Figure 2: An example of Proposition 2.3 for the graph of Figure 1, where  $T = \{e_2, e_3, e_5, e_7, e_8\}$  with the root  $v_1$ .

Proposition 2.3 suggests a simple way to compute  $\langle F \rangle_{\psi, v}$ , in analogy with the fact that a cycle space of a graph is spanned by fundamental cycles. For a connected  $X \subseteq E(G)$ , take a spanning tree  $T$  of the edge induced graph  $(V(X), X)$ . By Proposition 2.3 we can convert the labeling so that  $\psi(e) = \text{id}$  for all  $e \in T$ . Then, observe that any closed walk  $W \in \pi(X, v)$  can be considered as concatenations of closed walks  $W_1, \dots, W_k$  in  $\pi(X, v)$  such that  $W_i$  passes through an edge of  $X \setminus T$  only once. By  $\psi(e) = \text{id}$  for all  $e \in T$ , we deduce that  $\psi(W)$  is a product of elements in  $\{\psi(e) : e \in X \setminus T\}$ , where each  $\psi(e)$  is contained in  $\langle X \rangle_{\psi, v}$ . Thus, we have the following fact.

**Proposition 2.4.** *For a connected  $X \subseteq E(G)$  and a spanning tree  $T$  of  $(V(X), X)$ , suppose that  $\psi(e) = \text{id}$  for all  $e \in T$ . Then,  $\langle X \rangle_{\psi, v} = \langle \psi(e) \mid e \in X \setminus T \rangle$ .*

A connected edge subset  $F \subseteq E(G)$  is called *balanced* if  $\langle F \rangle_{\psi, v} = \text{id}$  for some  $v \in V(F)$ .  $F$  is called *unbalanced* if it is not balanced. By Proposition 2.1, this property is invariant under the choice of the base vertex  $v \in V(F)$ , and  $F$  is unbalanced if and only if  $F$  contains an unbalanced cycle. Thus, we can extend this notion to any  $F \subseteq E(G)$  (possibly disconnected sets) such that  $F$  is *unbalanced* if and only if  $F$  contains an unbalanced cycle.

### 3 Matroids and Polymatroids

#### 3.1 Polymatroids

Let  $E$  be a finite set. A function  $\mu : 2^E \rightarrow \mathbb{R}$  is called *submodular* if  $\mu(X) + \mu(Y) \geq \mu(X \cup Y) + \mu(X \cap Y)$  for every  $X, Y \subseteq E$ . It is well known that  $\mu : 2^E \rightarrow \mathbb{R}$  is submodular

if and only if  $\mu(X \cup \{e\}) - \mu(X) \geq \mu(Y \cup \{e\}) - \mu(Y)$  for any  $X \subseteq Y \subseteq E$  and  $e \in E \setminus Y$ .  $\mu$  is called *monotone* if  $\mu(X) \leq \mu(Y)$  for any  $X \subseteq Y$ .  $\mu$  is called *normalized* if  $\mu(\emptyset) = 0$ .

Suppose that  $\mu : 2^E \rightarrow \mathbb{Z}$  is a normalized integer-valued function on  $E$ . The pair  $(E, \mu)$  is called a (*integer*) *polymatroid* if  $\mu$  is monotone and submodular, and  $\mu$  is called the *rank function* of  $(E, \mu)$ .  $(E, \mu)$  is called a *matroid* if  $\mu$  further satisfies  $\mu(e) \leq 1$  for every  $e \in E$ .

### 3.2 Matroids induced by submodular functions

An integer-valued monotone submodular function  $\mu : 2^E \rightarrow \mathbb{Z}$  *induces* a matroid on  $E$ , denoted by  $\mathbf{M}(\mu)$ , where  $F \subseteq E$  is independent if and only if  $|X| \leq \mu(X)$  for every nonempty  $X \subseteq F$  [9]. This matroid can be understood through the following two polymatroid constructions, *Dilworth truncation* and *restriction*.

#### 3.2.1 Dilworth truncation

Let us first assume that  $\mu(e) \geq 0$  for every  $e \in E$ , and consider

$$\hat{\mu}(F) = \max\left\{\sum_{e \in F} x(e) \mid x \in \mathbb{R}_+^F : \emptyset \neq X \subseteq F, \sum_{e \in X} x(e) \leq \mu(X)\right\} \quad (2)$$

for  $F \subseteq E$ . It is known that  $\hat{\mu} : 2^E \rightarrow \mathbb{R}$  is a monotone submodular function, written by

$$\hat{\mu}(F) = \min\left\{\sum_{1 \leq i \leq k} \mu(F_i) \mid \text{a partition } \{F_1, \dots, F_k\} \text{ of } F\right\} \quad (F \subseteq E), \quad (3)$$

where  $\hat{\mu}(\emptyset) = 0$  (see, e.g., [29, Section 48.2] or [12, Theorem 2.6]). It is easy to check that, even if  $\mu(e) < 0$  holds,  $\hat{\mu}$  can be extended to be monotone submodular as follows:

$$\hat{\mu}(F) = \min\left\{\sum_{1 \leq i \leq k} \mu(F_i) \mid \text{a partition } \{F_1, \dots, F_k\} \text{ of } F_+\right\} \quad (F \subseteq E) \quad (4)$$

where  $F_+ = \{e \in F \mid \mu(e) \geq 0\}$ , and  $\hat{\mu}(F) = 0$  if  $F_+ = \emptyset$ . Since  $\hat{\mu}$  is non-negative,  $(E, \hat{\mu})$  is a polymatroid, which is called a *polymatroid induced by  $\mu$* , denoted  $\mathbf{P}(\mu)$ .  $\hat{\mu}$  is called the *Dilworth truncation* (or *lower truncation*) of  $\mu$  in the literature. See e.g., [29, 11, 12] for more detail on Dilworth truncations and applications.

#### 3.2.2 Restriction

Another important operation we will use is the *restriction* of  $\mu$  (to the hypercube). For polymatroid  $(E, \mu)$  (where  $\mu(\emptyset) = 0$  by definition), let  $\mu^1 : 2^E \rightarrow \mathbb{R}$  be

$$\mu^1(F) = \min\{|F \setminus X| + \mu(X) \mid X \subseteq F\} \quad (F \subseteq E). \quad (5)$$

Then, it can be seen that  $\mu^1$  is a monotone submodular function with  $\mu^1(F) \leq |F|$  (see e.g., [12, Section 3.1(b)]). In particular,  $\mu^1(e) \leq 1$  for every  $e \in E$ , which implies that  $(E, \mu^1)$  is a matroid if  $\mu$  is integer-valued. It is easy to see that  $F \subseteq E$  is independent in  $(E, \mu^1)$  if and only if  $|X| \leq \mu(X)$  holds for any  $X \subseteq F$ .

### 3.2.3 Rank formula of induced matroids

Combining these two operations, we now check that  $\hat{\mu}^1$  (i.e.,  $(\hat{\mu})^1$ ) is the rank of the matroid induced by an integer-valued monotone submodular function  $\mu$ . Note that, by (4) and (5),

$$\hat{\mu}^1(F) = \min \left\{ |F_+ \setminus \bigcup_{i=1}^k F_i| + \sum_{i=1}^k \mu(F_i) \mid \text{a subpartition } \{F_1, \dots, F_k\} \text{ of } F_+ \right\}, \quad (6)$$

and  $(E, \hat{\mu}^1)$  is a matroid. Since  $(E, \hat{\mu}^1)$  is obtained from  $(E, \hat{\mu})$  by a restriction,  $F \subseteq E$  is independent in  $(E, \hat{\mu}^1)$  if and only if  $|X| \leq \hat{\mu}(X)$  holds for any  $X \subseteq F$ . The latter condition is equivalent to  $|X| \leq \mu(X)$  for any non-empty  $X \subseteq F$  by (2). We thus have  $\mathbf{M}(\mu) = (E, \hat{\mu}^1)$ .

### 3.3 Matroid union

Let us consider two monotone submodular functions  $\mu_1$  and  $\mu_2$  on a finite set  $E$ . Since the monotonicity and the submodularity are preserved by taking summation,  $\mu_1 + \mu_2$  is monotone and submodular. Thus, for two polymatroids  $\mathbf{P}_1 = (E, \mu_1)$  and  $\mathbf{P}_2 = (E, \mu_2)$ ,  $(E, \mu_1 + \mu_2)$  forms a polymatroid, which is called the *sum* of  $\mathbf{P}_1$  and  $\mathbf{P}_2$ .

In a similar manner, suppose that we have two matroids  $\mathbf{M}_1 = (E, r_1)$  and  $\mathbf{M}_2 = (E, r_2)$  with the rank functions  $r_1$  and  $r_2$ . Their *union*  $\mathbf{M}_1 \vee \mathbf{M}_2$  is defined by  $(E, (r_1 + r_2)^1)$ , i.e.,  $(r_1 + r_2)^1(F) = \min\{|F \setminus X| + r_1(X) + r_2(X) \mid X \subseteq F\}$  for  $F \subseteq E$ . It is well known that  $F$  is independent in  $\mathbf{M}_1 \vee \mathbf{M}_2$  if and only if  $F$  can be partitioned into  $F_1$  and  $F_2$  such that  $F_i$  is independent in  $\mathbf{M}_i$  for  $i = 1, 2$  [8].

### 3.4 Linear Polymatroids

Let  $\mathbb{K}$  be a field and  $\mathbb{F}$  be a subfield of  $\mathbb{K}$  as defined in introduction. For a finite set  $E$ , let us associate a linear subspace  $A_e$  of  $\mathbb{F}^d$  with each  $e \in E$  by  $\Phi : e \in E \mapsto A_e \subseteq \mathbb{F}^d$ . Then,  $\dim_\Phi : 2^E \rightarrow \mathbb{Z}$ , defined by  $\dim_\Phi(F) = \dim(\text{span}\{A_e \mid e \in F\})$ , is a set function on  $E$ , and  $(E, \dim_\Phi)$  forms a polymatroid, denoted  $\mathbf{LP}(E, \Phi)$ . If a polymatroid  $(E, \mu)$  is isomorphic to  $\mathbf{LP}(E, \Phi)$  for some  $\Phi$  (i.e.,  $\mu(F) = \dim_\Phi(F)$  for any  $F \subseteq E$ ),  $(E, \mu)$  is said to be a *linear polymatroid*, and  $\Phi$  is called a *linear representation* of  $(E, \mu)$ .

If  $(E, \mu)$  is a matroid, a linear representation  $\Phi$  is sometimes referred to as an assignment of a vector, rather than a 1-dimensional linear space, with each element in  $E$ .

#### 3.4.1 Generic linear matroids

In §3.2, we have reviewed two polymatroid operations, restrictions and Dilworth truncations. Below, we shall take a look at geometric interpretations of these operations for linear polymatroids.

Let  $\mathbf{LP}(E, \Phi)$  be a linear polymatroid with a linear representation  $\Phi : e \in E \mapsto A_e \subseteq \mathbb{F}^d$ . For each  $e \in E$ , we shall pick a basis  $v_1, \dots, v_{k_e}$  of  $A_e$ , where  $k_e = \dim(A_e)$ , and define a *representative vector* by  $x_e = \sum_i \alpha_e^i v_i$ , where  $\alpha_e^i$  is a number in  $\mathbb{K}$  such that  $\{\alpha_e^i : e \in E, 1 \leq i \leq k_e\}$  is algebraically independent over  $\mathbb{F}$ . Then, extending the underlying field from  $\mathbb{F}$  to  $\mathbb{K}$ , we have *generically* chosen a representative vector  $x_e$  from each  $A_e$ .

This gives us a linear matroid with a linear representation  $e \mapsto x_e$  over  $\mathbb{K}$ . Lovász [18] gave its rank formula.



**Theorem 3.1** (Lovász [18]). *Let  $\mathbb{K}$  be a field and  $\mathbb{F}$  be a subfield of  $\mathbb{K}$ . Let  $\mathbf{LP}(E, \Phi)$  be a linear polymatroid with a linear representation  $\Phi : e \in E \mapsto A_e \subseteq \mathbb{F}^d$ , and suppose that a representative vector  $x_e$  is generically chosen from each  $A_e$  over  $\mathbb{K}$ . Then, over  $\mathbb{K}$ , we have the following:*

$$\dim(\text{span}\{x_e \mid e \in E\}) = \min\{|E \setminus F| + \dim(\text{span}\{A_e \mid e \in F\}) \mid F \subseteq E\}. \quad (7)$$

Note that the right hand side of (7) does not rely on the choice of representative vectors, and hence this motivates us to define the *generic matroid*. The *generic matroid obtained from  $\mathbf{LP}(E, \Phi)$* , denoted  $\mathbf{LM}(E, \Phi)$ , is defined to be a matroid with a linear representation  $e \mapsto x_e$  over  $\mathbb{K}$ . Notice the coincidence of two formula (5) and (7). Namely, taking the generic matroid is the same meaning as the restriction for linear polymatroids.

Lovász actually proved Theorem 3.1 under a much weaker assumption. For a family  $\{A_e \mid e \in E\}$  of linear subspaces in  $\mathbb{K}^d$ , a set of vectors  $x_e$  taken from each  $A_e$  is said to be in *generic position* if

$$x_f \in \text{span}\{x_e \mid e \in X\} \Rightarrow A_f \subseteq \text{span}\{x_e \mid e \in X\} \quad \forall X \subseteq E, \forall f \in E \setminus X \quad (8)$$

If  $\{x_e \mid e \in E\}$  is in generic position, (7) holds [18].

### 3.4.2 Dilworth truncation

We also have a geometric interpretation of Dilworth truncation. For a linear polymatroid  $\mathbf{LP} = (E, \Phi)$  with  $\Psi : e \mapsto A_e$ , let  $\mathcal{A} = \{A_e \mid e \in E\}$ . We now consider restricting  $\mathcal{A}$  to a generic hyperplane (i.e., a  $d - 1$  dimensional linear subspace) by extending the underlying field  $\mathbb{F}$  to  $\mathbb{K}$ , again. A hyperplane  $H$  is called *generic* if it is expressed by  $H = \{x \in \mathbb{K}^d \mid \sum_{1 \leq i \leq d} \alpha_i x(i) = 0\}$  for some algebraically independent numbers  $\{\alpha_1, \dots, \alpha_d\}$  over  $\mathbb{F}$ . Lovász [18] observed the following formula.

**Theorem 3.2** (Lovász [18]). *Let  $\mathbb{K}$  be a field and  $\mathbb{F}$  be a subfield of  $\mathbb{K}$ . Let  $\mathbf{LP}(E, \Phi)$  be a linear polymatroid with a linear representation  $\Phi : e \in E \mapsto A_e \subseteq \mathbb{F}^d$ , and  $H$  be a generic hyperplane of  $\mathbb{K}^d$ . Then, over  $\mathbb{K}$ , we have the following:*

$$\dim\{A_e \cap H \mid e \in E\} = \min\{\sum_{i=1}^k (\dim\{A_e \mid e \in E_i\} - 1)\}, \quad (9)$$

where the minimum is taken over all partitions  $\{E_1, \dots, E_k\}$  of  $E$  into nonempty subsets.

The same result was also obtained by Mason [23, 22] from the view point of combinatorial geometry (projective matroids), see also [5].

Setting  $\mu(F) = \dim\{A_e \mid e \in F\} - 1$  for  $F \subseteq E$ , we see that the polymatroid induced by  $\mu$ , that is  $(E, \hat{\mu})$ , has the linear representation  $e \mapsto A_e \cap H$  from the coincidence of (3) and (9).

### 3.4.3 Linear matroid union

A linear representation of the sum of two polymatroids can be easily obtained in the following manner. Suppose we have two linear polymatroids  $(E, \mu)$  and  $(E, \mu')$  with linear representations  $\Phi : e \in E \mapsto A_e \subseteq \mathbb{F}^a$  and  $\Phi' : e \in E \mapsto A'_e \subseteq \mathbb{F}^b$ , respectively. By definition,  $(\mu + \mu')(F) = \mu(F) + \mu'(F) = \dim\{A_e \mid e \in F\} + \dim\{A'_e \mid e \in F\}$ . Hence, if we

prepare  $\mathbb{F}^{a+b}$  as the underlying vector space, the polymatroid  $(E, \mu + \mu')$  is represented by  $e \mapsto A_e \oplus A'_e$ .

Combining this with the discussions of §3.3 and §3.4.1, it is now straightforward to see the following.

**Proposition 3.3.** *Let  $\mathbf{M}_i$  be a matroid on a finite set  $E$  with a linear representation  $e \mapsto x_e^i$  in a vector space  $W_i$  over  $\mathbb{F}$  for each  $i = 1, 2$ . Then,  $\mathbf{M}_1 \vee \mathbf{M}_2$  is represented by  $e \mapsto x_e$ , where  $x_e$  is a representative vector taken from  $\text{span}\{x_e^1\} \oplus \text{span}\{x_e^2\} \in W_1 \oplus W_2$  over  $\mathbb{K}$  in generic position.*

This fact is at least known from [22]. More detailed descriptions with examples can be found in [22, 23, 5].

## 4 Matroids Induced by Submodular Functions on Groups

### 4.1 Gain matroids

Let  $\Theta$  be the graph with two vertices  $u$  and  $v$  and three parallel edges. A subdivision of  $\Theta$  is called a theta graph. Hence, a theta graph consists of three openly disjoint paths between  $u$  and  $v$  and contains three cycles.

Consider an undirected multigraph, which may contain loops and parallel edges. A family  $\mathcal{C}$  of cycles is called a *linear class* if it satisfies the following property: if two cycles in  $\mathcal{C}$  form a theta subgraph, then the third cycle of the theta subgraph is also contained in  $\mathcal{C}$ . For a graph  $G = (V, E)$  and a linear class  $\mathcal{C}$  of cycles, the *biased matroid*  $\mathbf{B}(G, \mathcal{C})$  is defined such that  $F \subseteq E$  is independent if and only if each connected component of  $F$  contains no cycle or just one cycle, which is not included in the linear class  $\mathcal{C}$  [42, 43]. Therefore, the rank of  $F \subseteq E$  in  $\mathbf{B}(G, \mathcal{C})$  is equal to

$$g_{\mathcal{C}}(F) := |V(F)| - c(F) + \sum_{X \in \mathcal{C}(F)} \alpha_{\mathcal{C}}(X) \quad (F \subseteq E) \quad (10)$$

where

$$\alpha_{\mathcal{C}}(X) = \begin{cases} 1 & \text{if } X \text{ contains a cycle not included in } \mathcal{C} \\ 0 & \text{otherwise.} \end{cases}$$

This also implies that  $g_{\mathcal{C}}$  is monotone and submodular.

In this paper we are interested in biased matroids on gain graphs. Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph for a group  $\Gamma$  with a gain function  $\psi$ . Recall that a cycle is called balanced if the total gain walking through the cycle is equal to identity. This property does not depend on the choice of a base vertex of the walk by Proposition 2.1, and hence it is indeed a property of each cycle. Let  $\mathcal{C}$  be the set of balanced cycles in  $(G, \psi)$ . Then,  $\mathcal{C}$  forms a linear class, and the associated biased matroid is defined. This matroid is called the *gain matroid* of  $(G, \psi)$  [42, 43], denoted  $\mathbf{G}(G, \psi)$ . If we define  $g_{\Gamma} : 2^E \rightarrow \mathbb{Z}$  by

$$g_{\Gamma}(F) = |V(F)| - c(F) + \sum_{X \in \mathcal{C}(F)} \alpha_{\Gamma}(X) \quad (F \subseteq E) \quad (11)$$

where

$$\alpha_{\Gamma}(X) = \begin{cases} 1 & \text{if } X \text{ is unbalanced} \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

then we have  $\mathbf{G}(G, \psi) = (E, g_\Gamma)$ .

For a positive integer  $d$ , the union of  $d$  copies of  $\mathbf{G}(G, \psi)$  is  $(E, (dg_\Gamma)^1)$ . Namely, it is the matroid induced by

$$dg_\Gamma(F) = d|V(F)| - dc(F) + \sum_{X \in C(F)} d\alpha_\Gamma(X) \quad (F \subseteq E). \quad (13)$$

## 4.2 Lifting based on submodular functions on groups

We now extend the construction of the union of gain matroids by using structures of the underlying group. The idea is to replace the term  $\alpha_\Gamma$  by a function taking fractional values.

For a group  $\Gamma$ , we consider a function  $\mu : 2^\Gamma \rightarrow \mathbb{R}_+$  satisfying the following properties:

**(Normalized)**  $\mu(\emptyset) = 0$ ;

**(Monotonicity)**  $\mu(X) \leq \mu(Y)$  for any  $X \subseteq Y \subseteq \Gamma$ ;

**(Submodularity)**  $\mu(X) + \mu(Y) \geq \mu(X \cup Y) + \mu(X \cap Y)$  for any  $X, Y \subseteq \Gamma$ ;

**(Invariance under closures)**  $\mu(X) = \mu(\langle X \rangle)$  for any nonempty  $X \subseteq \Gamma$ ;

**(Invariance under conjugates)**  $\mu(X) = \mu(\gamma X \gamma^{-1})$  for any nonempty  $X \subseteq \Gamma$  and  $\gamma \in \Gamma$ .

We say that  $\mu : 2^\Gamma \rightarrow \mathbb{R}_+$  is a *symmetric polymatroidal function* over  $\Gamma$  if  $\mu$  satisfies these five conditions. The submodularity implies that, for any  $X \subseteq Y \subseteq \Gamma$  and  $e \in \Gamma$ ,

$$\mu(X \cup \{e\}) - \mu(X) \geq \mu(Y \cup \{e\}) - \mu(Y). \quad (14)$$

Extending the rank function (11) of gain matroids, we now propose a submodular function based on a symmetric polymatroidal function  $\mu$ . Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph. We consider  $\mu(\langle F \rangle_{\psi, v})$  for a connected  $F \subseteq E$  and  $v \in V(F)$ . By Proposition 2.1,  $\langle F \rangle_{\psi, v}$  is conjugate to  $\langle F \rangle_{\psi, u}$  for any  $u, v \in V(F)$  for  $F \subseteq E$ , and hence  $\mu(\langle F \rangle_{\psi, u}) = \mu(\langle F \rangle_{\psi, v})$  for any  $u, v \in V(F)$ . Also, by Proposition 2.2,  $\mu(\langle F \rangle_{\psi, v})$  is invariant with respect to the choice of equivalent gain functions  $\psi$ . We hence simply denote  $\mu(\langle F \rangle_{\psi, v})$  by  $\mu\langle F \rangle$ , implicitly assuming the gain function and the base vertex of the underlying homology among  $V(F)$ . We can then define a set function  $g_\mu : 2^E \rightarrow \mathbb{R}$  by

$$g_\mu(F) = |V(F)| - c(F) + \sum_{X \in C(F)} \mu\langle X \rangle \quad (F \subseteq E). \quad (15)$$

Notice that, if  $X$  and  $Y$  are connected with  $X \subseteq Y \subseteq E$ , we have  $\mu\langle X \rangle \leq \mu\langle Y \rangle$  by the monotonicity of  $\mu$  over  $\Gamma$ . However, the monotonicity and the submodularity of  $\mu$  do not hold over  $E$  in general. The next theorem ensures these properties for  $g_\mu$ , when  $\mu \leq 1$ .

**Theorem 4.1.** *Let  $\mu : 2^\Gamma \rightarrow [0, 1]$  be a symmetric polymatroidal function over a group  $\Gamma$  (with the upper bound 1), and  $(G = (V, E), \psi)$  a  $\Gamma$ -gain graph. Then,  $g_\mu$  is a monotone submodular function over  $E$ .*

*Proof.* For each  $X \subseteq E$  and  $e = (i, j) \in E \setminus X$ , let  $\Delta(X, e) = g_\mu(X \cup \{e\}) - g_\mu(X)$ . Let  $X_i$  be a connected component of  $X$  for which  $i \in V(X_i)$ . If such a component does not exist,

let  $X_i = \emptyset$ . Similarly, let  $X_j$  be a component of  $X$  for which  $j \in V(X_j)$ , where  $X_i = X_j$  if  $e$  is a loop.

By a simple calculation, we have the following relation:

$$\Delta(X, e) = \begin{cases} \mu\langle X_i \cup \{e\} \rangle - \mu\langle X_i \rangle & \text{if } X_i = X_j \\ \mu\langle X_i \cup X_j \cup \{e\} \rangle + 1 - \mu\langle X_i \rangle - \mu\langle X_j \rangle & \text{otherwise.} \end{cases} \quad (16)$$

Let us check the monotonicity. Suppose that  $X_i = X_j$ . Due to the monotonicity of  $\mu$  over  $\Gamma$ ,  $\mu\langle X_i \cup \{e\} \rangle - \mu\langle X_i \rangle \geq 0$ . On the other hand, suppose that  $X_i \neq X_j$ . Since  $X_i$  and  $X_i \cup X_j \cup \{e\}$  are connected, we have  $\mu\langle X_i \rangle \leq \mu\langle X_i \cup X_j \cup \{e\} \rangle$  by the monotonicity of  $\mu$  over  $\Gamma$ . Also, by the upper bound of  $\mu$ ,  $\mu\langle X_j \rangle \leq 1$ . We thus have  $\Delta(X, e) = \mu\langle X_i \cup X_j \cup \{e\} \rangle + 1 - (\mu\langle X_i \rangle + \mu\langle X_j \rangle) \geq 0$ . This completes the proof of the monotonicity.

For the submodularity, we check

$$\Delta(X, e) \geq \Delta(Y, e) \quad (17)$$

for any  $X \subseteq Y \subseteq E$  and  $e \in E \setminus Y$ . We split the proof into two cases depending on whether  $X_i = X_j$  or not.

Case 1: Suppose  $X_i = X_j$ . We then have  $X_i \subseteq Y_i = Y_j$ . We take a tree  $T \subseteq Y_i$  spanning  $V(Y_i)$  such that  $T \cap X_i$  forms a tree spanning  $V(X_i)$ . By using switching operations, we may assume by Proposition 2.3 that  $\psi(f) = \text{id}$  for every  $f \in T$ . Observe then that  $\langle Y_i \cup \{e\} \rangle_{\psi, i} = \langle \langle Y_i \rangle_{\psi, i} \cup \psi(e) \rangle$  and  $\langle X_i \cup \{e\} \rangle_{\psi, i} = \langle \langle X_i \rangle_{\psi, i} \cup \psi(e) \rangle$  by Proposition 2.4. We thus have

$$\begin{aligned} \Delta(X, e) &= \mu\langle X_i \cup \{e\} \rangle - \mu\langle X_i \rangle \\ &= \mu(\langle \langle X_i \rangle_{\psi, i} \cup \{ \psi(e) \} \rangle) - \mu(\langle X_i \rangle_{\psi, i}) \\ &= \mu(\langle X_i \rangle_{\psi, i} \cup \{ \psi(e) \}) - \mu(\langle X_i \rangle_{\psi, i}) \\ &\geq \mu(\langle Y_i \rangle_{\psi, i} \cup \{ \psi(e) \}) - \mu(\langle Y_i \rangle_{\psi, i}) \\ &= \mu(\langle \langle Y_i \rangle_{\psi, i} \cup \{ \psi(e) \} \rangle) - \mu(\langle Y_i \rangle_{\psi, i}) \\ &= \mu\langle Y_i \cup \{e\} \rangle - \mu\langle Y_i \rangle = \Delta(Y, e), \end{aligned}$$

where we used (14), (16), and the invariance of  $\mu$  under closures.

Case 2. Suppose that  $X_i \neq X_j$ . We further split the proof into subcases:

(2-i) If  $Y_i = Y_j$ , then, by (16), we have  $\Delta(X, e) - \Delta(Y, e) = \mu\langle X_i \cup X_j \cup \{e\} \rangle + 1 + \mu\langle Y_i \rangle - \mu\langle X_i \rangle - \mu\langle X_j \rangle - \mu\langle Y_i \cup \{e\} \rangle$ . Since all these sets are connected,  $\mu\langle X_i \cup X_j \cup \{e\} \rangle \geq \mu\langle X_j \rangle$ ,  $\mu\langle Y_i \rangle \geq \mu\langle X_i \rangle$ , and  $1 \geq \mu\langle Y_i \cup \{e\} \rangle$ . Thus,  $\mu\langle X_i \cup X_j \cup \{e\} \rangle + \mu\langle Y_i \rangle + 1 \geq \mu\langle X_i \rangle + \mu\langle X_j \rangle + \mu\langle Y_i \cup \{e\} \rangle$ , implying (17).

(2-ii) If  $Y_i \neq Y_j$ , we have  $X_i \subseteq Y_i$  and  $X_j \subseteq Y_j$ . Then,  $e$  is a bridge connecting  $X_i$  and  $X_j$  in  $X_i \cup X_j \cup \{e\}$  and is also a bridge connecting  $Y_i$  and  $Y_j$  in  $Y_i \cup Y_j \cup \{e\}$ . By a switch operation, we may assume  $\psi(e) = \text{id}$ . Then,  $\langle X_i \cup X_j \cup \{e\} \rangle_{\psi, i} = \langle \langle X_i \rangle_{\psi, i} \cup \langle X_j \rangle_{\psi, j} \rangle$ . This implies  $\mu\langle X_i \cup X_j \cup \{e\} \rangle = \mu(\langle X_i \rangle_{\psi, i} \cup \langle X_j \rangle_{\psi, j})$  by the invariance under closures. Symmetrically, we have  $\mu\langle Y_i \cup Y_j \cup \{e\} \rangle = \mu(\langle Y_i \rangle_{\psi, i} \cup \langle Y_j \rangle_{\psi, j})$ . By using the submodularity

and the monotonicity of  $\mu$  over  $\Gamma$ , along with  $X_k \subseteq Y_k$  for  $k = 1, 2$ , we have

$$\begin{aligned}
& \mu\langle X_i \cup X_j \cup \{e\} \rangle + \mu\langle Y_i \rangle + \mu\langle Y_j \rangle \\
&= \mu(\langle X_i \rangle_i \cup \langle X_j \rangle_j) + \mu(\langle Y_i \rangle_i) + \mu(\langle Y_j \rangle_j) \\
&\geq \mu(\langle X_i \rangle_i \cup \langle X_j \rangle_j \cup \langle Y_i \rangle_i) + \mu((\langle X_i \rangle_i \cup \langle X_j \rangle_j) \cap \langle Y_i \rangle_i) + \mu(\langle Y_j \rangle_j) \\
&\geq \mu(\langle Y_i \rangle_i \cup \langle X_j \rangle_j) + \mu(\langle X_i \rangle_i) + \mu(\langle Y_j \rangle_j) \\
&\geq \mu(\langle Y_i \rangle_i \cup \langle X_j \rangle_j \cup \langle Y_j \rangle_j) + \mu((\langle Y_i \rangle_i \cup \langle X_j \rangle_j) \cap \langle Y_j \rangle_j) + \mu(\langle X_i \rangle_i) \\
&\geq \mu(\langle Y_i \rangle_i \cup \langle Y_j \rangle_j) + \mu(\langle X_j \rangle_j) + \mu(\langle X_i \rangle_i) \\
&= \mu\langle Y_i \cup Y_j \cup \{e\} \rangle + \mu\langle X_j \rangle + \mu\langle X_i \rangle.
\end{aligned}$$

This implies (17) by (16).  $\square$

The aim of this paper is to extend the concept of the union of gain matroids. We shall thus concentrate only on a function  $\mu$  taking fractional values, that is,  $\mu : 2^\Gamma \rightarrow \{0, \frac{1}{d}, \dots, \frac{d-1}{d}, 1\}$  for some finite positive integer  $d$ . As it is not integer-valued,  $g_\mu$  does not induce a matroid in general, but if we define  $f_\mu : 2^E \rightarrow \mathbb{Z}$  by

$$f_\mu(F) = dg_\mu(F) \quad (F \subseteq E), \quad (18)$$

$f_\mu$  is a normalized integer-valued monotone submodular function by Theorem 4.1; thus,  $f_\mu$  induces a polymatroid  $\mathbf{P}(f_\mu) = (E, f_\mu)$  and a matroid  $\mathbf{M}(f_\mu) = (E, f_\mu^1)$  on  $E$ . We will see special classes of  $\mathbf{M}(f_\mu)$  in several applications.

*Example 4.1.* The gain matroid (or the union of copies) is a special case of  $\mathbf{M}(f_\mu)$ , where  $\mu$  is defined by  $\mu(X) = 0$  for  $X = \emptyset$  or  $X = \text{id}$ , and otherwise  $\mu(X) = 1$ . In this case, (18) is equal to (13).  $\square$

*Example 4.2.* Let us consider a group  $\Gamma$  equipped with a linear representation  $\rho : \Gamma \rightarrow GL(\mathbb{F}^d)$  over a field  $\mathbb{F}$ . Let  $d_\rho : 2^\Gamma \rightarrow \mathbb{Z}$  be a function defined by

$$d_\rho(X) = \dim(\text{span}\{\text{im}(I_d - \rho(\gamma)) \mid \gamma \in X\}) \quad (X \subseteq \Gamma),$$

where  $d_\rho(\emptyset) = 0$  and  $I_d$  denotes the identity matrix of size  $d \times d$ .

It is easy to see that  $d_\rho$  is monotone submodular and is invariant under conjugates. Also, for any  $\gamma_1, \gamma_2 \in \Gamma$ , we have  $\text{im}(I_d - \rho(\gamma_1\gamma_2)) \subseteq \text{im}(I_d - \rho(\gamma_1)) + \text{im}(I_d - \rho(\gamma_2))$  since  $(I_d - \rho(\gamma_1))(I_d - \rho(\gamma_2)) = -(I_d - \rho(\gamma_1\gamma_2)) + (I_d - \rho(\gamma_1)) + (I_d - \rho(\gamma_2))$  and  $\text{im}(I_d - \rho(\gamma_1))(I_d - \rho(\gamma_2)) \subseteq \text{im}(I_d - \rho(\gamma_1))$ . This implies the invariance of  $d_\rho$  under closures. Therefore, by setting  $\mu = d_\rho/d$ , we have another example of a symmetric polymatroidal function  $\mu$ . The corresponding matroid will be extensively discussed in the next section.  $\square$

## 5 Matroids Induced by Group Representations

Dowling geometries [7] are special cases of gain matroids for finite groups, which admit linear representations over finite fields  $\mathbb{F}$ . In this section, we shall extend the union of Dowling geometries based on the group representations.

## 5.1 Dowling Geometries

Suppose that  $\Gamma$  is a finite group and  $n$  is a positive integer. Define a  $\Gamma$ -gain graph  $(K_n^\bullet(\Gamma), \psi^\bullet)$  on  $V(K_n^\bullet(\Gamma)) = \{1, 2, \dots, n\}$  such that (i) for every  $i, j$  with  $1 \leq i < j \leq n$  and every  $\gamma \in \Gamma$ , it has an edge from  $i$  to  $j$  with the gain  $\gamma$  and (ii) for each vertex  $i$ , it has a loop attached to  $i$  with a gain  $\gamma_i$ , where  $\gamma_i$  is any non-identity element of  $\Gamma$  (assuming that  $\Gamma$  is nontrivial). The *Dowling geometry*  $\mathbf{D}_n(\Gamma)$  is defined by  $\mathbf{G}(K_n^\bullet(\Gamma), \psi^\bullet)$ , the gain matroid of  $(K_n^\bullet(\Gamma), \psi^\bullet)$ .

The remarkable property of  $\mathbf{D}_n(\Gamma)$  is that, for  $n \geq 3$ ,  $\mathbf{D}_n(\Gamma)$  is representable over  $\mathbb{F}$  if and only if  $\Gamma$  is isomorphic to a subgroup of  $\mathbb{F}^\times$  (see [17] or [25, Theorem 6.10.10]). The proof of one direction indicates the following explicit construction of representations.

Suppose that  $\Gamma$  is isomorphic to a subgroup of  $\mathbb{F}^\times$ . For a simpler description, we assume that  $\Gamma$  is itself a subgroup of  $\mathbb{F}^\times$ . With each  $e = (i, j) \in E(K_n^\bullet(\Gamma))$ , we associate a vector  $x_e \in \mathbb{F}^V$  defined by

$$x_e(v) = \begin{cases} -\psi(e) & \text{if } v = i \\ 1 & \text{if } v = j \\ 0 & \text{otherwise} \end{cases}$$

if  $e$  is not a loop, and

$$x_e(v) = \begin{cases} 1 & \text{if } v = i \\ 0 & \text{otherwise} \end{cases}$$

if  $e$  is a loop. These give us a linear representation of  $\mathbf{D}_n(\Gamma)$  over  $\mathbb{F}$  (see e.g., [25, Lemma 6.10.11]), which is called the *canonical representation* [44]. As each  $\Gamma$ -gain graph  $(G, \psi)$  can be considered as a subgraph of  $(K_n^\bullet(\Gamma), \psi^\bullet)$ , the restriction to  $E(G)$  leads to the canonical representation of  $\mathbf{G}(G, \psi)$ .

Equivalently, instead of a vector assignment, we may associate a 1-dimensional linear space

$$D_e = \left\{ x \in \mathbb{F}^V \mid \begin{array}{l} x(i) + \psi(e)x(j) = 0, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\} \quad (19)$$

with each non-loop edge  $e = (i, j)$ , and

$$D_e = \{x \in \mathbb{F}^V \mid x(V \setminus \{i\}) = 0\} \quad (20)$$

with a loop  $e$  attached to  $i$ , where, for  $W \subseteq V$ ,  $x(W) = 0$  implies  $x(k) = 0$  for all  $k \in W$ . Then, the union of  $d$  copies of  $\mathbf{D}_n(\Gamma)$  can be obtained in a systematic manner, by just following the technique mentioned in § 3.4.3.

Let us consider the direct sum of  $d$  copies of  $\mathbb{F}^V$ , which results in  $(\mathbb{F}^d)^V$ . Then, the associated vector space with each edge  $e = (i, j)$  becomes a  $d$ -dimensional space in  $(\mathbb{F}^d)^V$  written by

$$D_e^d = \left\{ x \in (\mathbb{F}^d)^V \mid \begin{array}{l} x(i) + \psi(e)x(j) = 0, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\} \quad (21)$$

and

$$D_e^d = \{x \in (\mathbb{F}^d)^V \mid x(V \setminus \{i\}) = 0\}. \quad (22)$$

By extending the underlying field from  $\mathbb{F}$  to  $\mathbb{K}$ , we shall take a representative vector  $x_e$  from each  $D_e^d$  in generic position. By Proposition 3.3, we obtain a linear representation

$\Psi : e = (i, j) \mapsto x_e^d \in D_e^d$  of the union of  $d$  copies of Dowling geometry  $\mathbf{D}_n(\Gamma)$ , each vector written by

$$x_e^d(v) = \begin{cases} -\psi(e)\alpha_e & \text{if } v = i \\ \alpha_e & \text{if } v = j \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

$$x_e^d(v) = \begin{cases} \alpha_e & \text{if } v = i \\ 0 & \text{otherwise,} \end{cases} \quad (24)$$

depending on whether  $e = (i, j)$  is a non-loop or a loop, where  $\alpha_e = (\alpha_e^1, \dots, \alpha_e^d)^\top \in \mathbb{K}^d$  such that  $\{\alpha_e^i : 1 \leq i \leq d, e \in E(K_n^\bullet(\Gamma))\}$  is algebraically independent over  $\mathbb{F}$ .

In the subsequent discussion, we need the following (more or less) known fact about graphic matroids. Consider a gain graph  $(G = (V, E), \psi)$  such that  $\psi(e) = \text{id}$  for every  $e \in E$ . Then,  $\mathbf{G}(G, \psi)$  is just the graphic matroid of  $G$ . The result of linear representations of gain matroids implicitly implies the following fact on the linear representation of the sum of  $d$  copies of the graphic matroid, which will be frequently used in the subsequent discussion.

**Lemma 5.1.** *Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph such that  $\psi(e) = \text{id}$  for every  $e \in E$ . Suppose that  $E$  is connected. Then, the following holds.*

- $\dim(\text{span}\{D_e^d \mid e \in E\}) = d|V| - d$ ;
- For any  $x \in (\mathbb{F}^d)^V$  with  $x(V \setminus \{i, j\}) = 0$ ,  $x \in \text{span}\{D_e^d \mid e \in E\}$  if and only if  $x(i) + x(j) = 0$ .

*Proof.* The first part directly follows from the above discussion on gain matroids and the canonical linear representation. Indeed, since  $D_e^d$  is the direct sum of  $d$  copies of  $D_e$ , we have  $\dim(\text{span}\{D_e^d \mid e \in E\}) = d \dim(\text{span}\{D_e \mid e \in E\}) = d(|V| - c(E) + \alpha_\Gamma(E)) = d|V| - d$ .

The second part also follows from the above discussion. We first consider the case of  $d = 1$ . Let  $f = (i, j)$  be a new edge from  $i$  to  $j$  with the gain  $\psi(f) = a$  for some  $a \in \mathbb{F} \setminus \{0\}$ . In the canonical representation of the gain matroid on  $E \cup \{f\}$ ,  $f$  is associated with a vector  $x \in \mathbb{F}^V$  with  $x(V \setminus \{i, j\}) = 0$ ,  $x(j) = 1$  and  $x(i) = -a$ .

Note that, since  $E$  is connected,  $E \cup \{f\}$  has a cycle passing through  $f$ . We thus have  $\langle E \cup \{f\} \rangle_{\psi, v} = \langle a \rangle$  for any  $v \in V$  by Proposition 2.4, since every gain of  $E$  is identity. This implies that

$$\dim(\text{span}\{D_e \mid e \in E \cup \{f\}\}) = g_\Gamma(E \cup \{f\}) = \begin{cases} |V| - 1 & \text{if } a = 1 \\ |V| & \text{otherwise.} \end{cases}$$

Therefore,  $\text{span}\{D_e \mid e \in E\}$  contains  $x \in \mathbb{F}^V \setminus \{0\}$  with  $x(V \setminus \{i, j\}) = 0$  and  $x(i) + ax(j) = 0$  if and only if  $a = 1$ . Equivalently,  $\text{span}\{D_e \mid e \in E\}$  contains  $x \in \mathbb{F}^V \setminus \{0\}$  with  $x(V \setminus \{i, j\}) = 0$  if and only if  $x(i) + x(j) = 0$ .

Consequently, if we consider the direct sum of  $d$  copies of each linear space, we conclude that  $\text{span}\{D_e^d \mid e \in E\}$  contains  $x \in (\mathbb{F}^d)^V \setminus \{0\}$  with  $x(V \setminus \{i, j\}) = 0$  if and only if  $x(i) + x(j) = 0$ .  $\square$

## 5.2 Linear matroids induced by group representations

In this section, we shall extend the representation theory of the union of Dowling geometries. The idea of our construction is that, instead of coefficients  $\psi(e) \in \mathbb{F}^\times$  of  $\alpha_e$  in (23), we shall make use of linear representations of groups. We then have a linear matroid induced by a group  $\Gamma$ , where  $\Gamma$  is not restricted to abelian groups. We show that resulting linear matroids are special cases of matroids given in §4, as in the relation between Dowling geometries and gain matroids.

Let  $\Gamma$  be a group equipped with a linear representation  $\rho : \Gamma \rightarrow GL(\mathbb{F}^d)$  on a vector space of finite dimension  $d$  over a field  $\mathbb{F}$ . Let  $(G = (V, E), \psi)$  be a finite  $\Gamma$ -gain graph.

As in the previous subsection, let  $\mathbb{K}$  be an extension of  $\mathbb{F}$  that contains an algebraically independent set  $\{\alpha_e^i \mid i = 1, \dots, d, e \in E\}$  over  $\mathbb{F}$ , and let  $\alpha_e = (\alpha_e^1, \dots, \alpha_e^d)^\top \in \mathbb{K}^d$ .

With each  $e = (i, j) \in E$ , we assign a vector  $x_{e,\psi} \in (\mathbb{K}^d)^V$  defined by

$$x_{e,\psi}(v) = \begin{cases} -\rho(\psi(e))\alpha_e & \text{if } v = i \\ \alpha_e & \text{if } v = j \\ 0 & \text{otherwise} \end{cases} \quad (25)$$

if  $e = (i, j)$  is not a loop, and

$$x_{e,\psi}(v) = \begin{cases} (I_d - \rho(\psi(e)))\alpha_e & \text{if } v = i \\ 0 & \text{otherwise,} \end{cases} \quad (26)$$

if  $e$  is a loop. The linear matroid induced on  $\{x_{e,\psi} \mid e \in E\}$  is denoted by  $\mathbf{D}_\rho(G, \psi)$ .

Note that  $\mathbf{D}_\rho(G, \psi)$  is the generic matroid obtained from a linear polymatroid with a linear representation  $e \mapsto A_{e,\psi}$  defined by

$$A_{e,\psi} = \left\{ x \in (\mathbb{F}^d)^V \mid \begin{array}{l} x(i) + \rho(\psi(e))x(j) = 0, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\} \quad (27)$$

for each non-loop edge  $e \in E$ , and

$$A_{e,\psi} = \left\{ x \in (\mathbb{F}^d)^V \mid \exists \alpha \in \mathbb{F}^d : \begin{array}{l} x(i) = (I_d - \rho(\psi(e)))\alpha, \\ x(V \setminus \{i\}) = 0 \end{array} \right\} \quad (28)$$

for a loop  $e$ , by extending the underlying field from  $\mathbb{F}$  to  $\mathbb{K}$ . Note that  $A_{e,\psi}$  is invariant with the choice of orientation of each edge, as each  $\rho(\psi(e))$  is invertible.

Let  $\mathbf{DP}_\rho(G, \psi)$  be the linear polymatroid on  $E$  represented by  $\Psi : e \mapsto A_{e,\psi}$ . Clearly, each  $A_{e,\psi}$  depends on the gain function  $\psi$ , but, as shown below, the associated polymatroid is actually invariant up to equivalent  $\psi$ .

**Lemma 5.2.** *Let  $\psi$  and  $\psi'$  be equivalent gain functions. Then,*

$$\dim(\text{span}\{A_{e,\psi} \mid e \in E\}) = \dim(\text{span}\{A_{e,\psi'} \mid e \in E\}).$$

*Proof.* Let us simply denote  $d_\psi = \dim\{\text{span}\{A_{e,\psi} \mid e \in E\}\}$ . It is sufficient to show that  $d_\psi$  is invariant from any switch operation.

Suppose that  $\psi'$  is obtained from  $\psi$  by a switch operation at  $v \in V$  with  $\gamma \in \Gamma$ . Since  $A_{e,\psi}$  is invariant with the choice of the edge orientation, we may assume that all of edges



are oriented from  $v$ . Then,  $\psi'(e) = \gamma\psi(e)$  if  $e$  is incident to  $v$ ,  $\psi'(e) = \gamma\psi(e)\gamma^{-1}$  if  $e$  is a loop at  $v$ , and otherwise  $\psi'(e) = \psi(e)$ .

Consider a bijective linear transformation  $T : (\mathbb{F}^d)^V \rightarrow (\mathbb{F}^d)^V$  defined by, for each  $x \in (\mathbb{F}^d)^V$ ,

$$T(x)(w) = \begin{cases} x(w) & \text{if } w \in V \setminus \{v\} \\ \rho(\gamma)x(v) & \text{if } w = v. \end{cases}$$

We then have

$$x(v) = \rho(\gamma)^{-1}T(x)(v) \quad \text{and} \quad x(w) = T(x)(w) \text{ for } w \in V \setminus \{v\}.$$

Therefore, if  $e$  is a non-loop edge oriented from  $v$  to a vertex  $j \in V$ ,

$$\begin{aligned} TA_{e,\psi} &= \{T(x) \in (\mathbb{F}^d)^V \mid x(v) + \rho(\psi(e))x(j) = 0, x(V \setminus \{v, j\}) = 0\} \\ &= \{y \in (\mathbb{F}^d)^V \mid y(v) + \rho(\gamma\psi(e))y(j) = 0, y(V \setminus \{v, j\}) = 0\}. \end{aligned}$$

As  $\psi'(e) = \gamma\psi(e)$ , we obtain that  $TA_{e,\psi} = A_{e,\psi'}$ . Similarly, if  $e$  is a loop attached to  $v$ ,

$$\begin{aligned} TA_{e,\psi} &= \{y \in (\mathbb{K}^d)^V \mid \exists \alpha \in \mathbb{K}^d: \rho(\gamma^{-1})y(v) = (I_d - \rho(\psi(e)))\alpha, y(V \setminus \{v\}) = 0\} \\ &= \{y \in (\mathbb{K}^d)^V \mid \exists \alpha \in \mathbb{K}^d: y(v) = (I_d - \rho(\gamma\psi(e)\gamma^{-1}))\alpha, y(V \setminus \{v\}) = 0\} \\ &= A_{e,\psi'}, \end{aligned}$$

where  $\psi'(e) = \gamma\psi(e)\gamma^{-1}$ .

If  $e$  is not incident to  $v$ , then we clearly have  $TA_{e,\psi} = A_{e,\psi} = A_{e,\psi'}$ . In total,  $d_\psi$  is invariant from any switch operation.  $\square$

By Lemma 5.2,  $\mathbf{DP}_\rho(G, \psi)$  is invariant with respect to the choice of equivalent gain functions  $\psi$ . Since the matroid  $\mathbf{D}_\rho(G, \psi)$  is obtained from  $\mathbf{DP}_\rho(G, \psi)$ , we also found that  $\mathbf{D}_\rho(G, \psi)$  is invariant up to equivalent gain functions  $\psi$ .

### 5.3 Combinatorial characterization

We now show that the linear matroid  $\mathbf{D}_\rho(G, \psi)$  is indeed equal to a special case of matroids given in §4.2. Recall that  $d_\rho : 2^\Gamma \rightarrow \mathbb{Z}$  is given in §4.2 (Example 4.2) by

$$d_\rho(X) = \dim(\text{span}\{\text{im}(I_d - \rho(\gamma)) \mid \gamma \in X\}) \quad (X \subseteq \Gamma). \quad (29)$$

Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph. As a special case of  $f_\mu$  given in (18), we shall define a set function on  $E$  by

$$f_\rho(F) = d|V(F)| - dc(F) + \sum_{X \in C(F)} d_\rho\langle X \rangle \quad (F \subseteq E). \quad (30)$$

By Theorem 4.1,  $f_\rho$  is a normalized integer-valued monotone submodular function, and thus  $\mathbf{P}(f_\rho) = (E, f_\rho)$  is a polymatroid and  $\mathbf{M}(f_\rho) = (E, f_\rho^1)$  is a matroid on  $E$ .

We are now ready to state our main theorems. The first theorem asserts the equivalence of  $\mathbf{P}(f_\rho)$  and  $\mathbf{DP}_\rho(G, \psi)$ , while the second implies the equivalence of  $\mathbf{M}(f_\rho)$  and  $\mathbf{D}_\rho(G, \psi)$ .

**Theorem 5.3.** *Let  $\Gamma$  be a group equipped with a linear representation  $\rho : \Gamma \rightarrow GL(\mathbb{F}^d)$ . Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph. Then,*

$$f_\rho(E) = \dim(\text{span}\{A_{e,\psi} \mid e \in E\}). \quad (31)$$

*Proof.* For any  $F \subseteq E$ , let  $A_F = \text{span}\{A_{e,\psi} \mid e \in F\}$ . By definition, it is easy to check that  $A_E = \bigoplus_{X \in C(E)} A_X$ . Moreover,  $f_\rho(E) = \sum_{X \in C(E)} f_\rho(X)$ . Hence, it suffices to show the statement when  $G$  is connected.

Let  $T$  be a spanning tree in  $E$ . By Proposition 2.3 and Lemma 5.2, we may assume that  $\psi(e) = I_d$  for  $e \in T$ . By Proposition 2.4,  $\langle X \rangle_{\psi,v} = \langle \psi(e) \mid e \in E \setminus T \rangle$  for any  $v \in V(X)$ . Hence, as  $\mu$  is invariant under the closures,  $\mu\langle X \rangle = \mu(\{\psi(e) \mid e \in E \setminus T\})$ . Thus,

$$f_\rho(E) = d|V| - d + \dim(\text{span}\{\text{im}(I_d - \rho(\psi(e))) \mid e \in E \setminus T\}).$$

By Lemma 5.1, we have that (i)  $\dim(A_T) = d|V| - d$  and (ii) for any  $i, j \in V$  and any  $x \in (\mathbb{F}^d)^V$  with  $x(V \setminus \{i, j\}) = 0$ ,  $x \in A_T$  if and only if  $x(i) + x(j) = 0$ . This means that each quotient space  $A_{e,\psi}/A_T$  for  $e \in E \setminus T$  is written by

$$\begin{aligned} A_{e,\psi}/A_T &= \{x + A_T \mid \exists \alpha \in \mathbb{F}^d: x(j) = (I_d - \rho(\psi(e)))\alpha, x(V \setminus \{j\}) = 0\} \\ &= \{x + A_T \mid \exists \alpha \in \mathbb{F}^d: x(1) = (I_d - \rho(\psi(e)))\alpha, x(V \setminus \{1\}) = 0\} \end{aligned}$$

where 1 denotes a vertex in  $V$ . Therefore,  $\text{span}\{A_{e,\psi}/A_T \mid e \in E \setminus T\}$  is isomorphic to

$$\text{span}\{x \in (\mathbb{F}^d)^V \mid \exists e \in E \setminus T, \exists \alpha \in \mathbb{F}^d: x(1) = (I_d - \rho(\psi(e)))\alpha, x(V \setminus \{1\}) = 0\}$$

which is isomorphic to  $\text{span}\{\text{im}(I_d - \rho(\psi(e))) \mid e \in E \setminus T\}$ . Consequently, we obtain

$$\dim(\text{span}\{A_{e,\psi}/A_T \mid e \in E \setminus T\}) = \dim(\text{span}\{\text{im}(I_d - \rho(\psi(e))) \mid e \in E \setminus T\}).$$

In total, we obtain  $\dim(\{A_{e,\psi} \mid e \in E\}) = \dim(A_T) + \dim(\text{span}\{A_{e,\psi}/A_T \mid e \in E \setminus T\}) = d|V| - d + \dim(\text{span}\{\text{im}(I_d - \rho(\psi(e))) \mid e \in E \setminus T\}) = f_\rho(E)$ , completing the proof of Theorem 5.3.  $\square$

**Theorem 5.4.** *Let  $\Gamma$  be a group equipped with a linear representation  $\rho : \Gamma \rightarrow GL(\mathbb{F}^d)$ . Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph. Then,  $f_\rho^1$  is the rank function of  $\mathbf{D}_\rho(G, \psi)$ .*

*Proof.* By Theorem 5.3, we have  $f_\rho(F) = \dim(\text{span}\{A_{e,\psi} \mid e \in F\})$  for any  $F \subseteq E$  (by restricting the statement to the graph  $(V, F)$ ). Since  $x_{e,\psi}$  of (25)(26) is taken from  $A_{e,\psi}$  so that  $\{x_{e,\psi} \mid e \in E\}$  is in generic position, Theorem 3.1 implies that the rank of  $F \subseteq E$  in  $\mathbf{D}_\rho(G)$  is equal to  $\min\{|X| + \dim(\text{span}\{A_{e,\psi} \mid e \in F \setminus X\}) \mid X \subseteq F\}$ , which is equal to  $f_\rho^1(F)$  by definition (5).  $\square$

**Corollary 5.5.** *Let  $\Gamma$  be a group equipped with a linear representation  $\rho : \Gamma \rightarrow GL(\mathbb{F}^d)$ . Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph. Then,  $\{x_{e,\psi} \mid e \in E\}$  is linearly independent if and only if  $|F| \leq d|V(F)| - dc(F) + \sum_{X \in C(F)} d_\rho\langle X \rangle$  for any  $F \subseteq E$ .*

## 6 Applications

As applications, we shall address two problems from discrete geometry; one is on the parallel redrawings of symmetric graphs and the other is on the rigidity of symmetric graphs.

We shall first introduce notions of symmetric graphs and symmetric frameworks in §6.1 and §6.2, respectively. Then, we will discuss the symmetric redrawing problem of symmetric frameworks in §6.3 and the symmetric rigidity of symmetric frameworks in §6.4.

## 6.1 Symmetric graphs

Let  $H$  be a simple graph, which may not be finite. An *automorphism* of  $H$  is a permutation  $\pi : V(H) \rightarrow V(H)$  such that  $\{u, v\} \in E(H)$  if and only if  $\{\pi(u), \pi(v)\} \in E(H)$ . The set of all automorphisms of  $H$  forms a subgroup of the symmetric group of  $V(H)$ , known as the *automorphism group*  $\text{Aut}(H)$  of  $H$ . An *action* of a group  $\Gamma$  on  $H$  is a group homomorphism  $\theta : \Gamma \rightarrow \text{Aut}(H)$ . An action  $\theta$  is called *free* if  $\theta(\gamma)(v) \neq v$  for any  $v \in V$  and any non-identity  $\gamma \in \Gamma$ . We say that a graph  $H$  is  $(\Gamma, \theta)$ -*symmetric* if  $\Gamma$  acts on  $H$  by  $\theta$ . In the subsequent discussion, we only consider free actions, and we omit to specify the action  $\theta$ , if it is clear from the context. We then denote  $\theta(\gamma)(v)$  by  $\gamma v$ .

For an  $\Gamma$ -symmetric graph  $H$ , the *quotient graph*  $H/\Gamma$  is a multigraph on the set  $V(H)/\Gamma$  of vertex orbits, together with the set  $E(H)/\Gamma$  of edge orbits as the edge set. An edge orbit may be represented by a loop in  $H/\Gamma$ . Figure 3 illustrates an example when  $\Gamma$  is the dihedral group of order 4.

Several distinct graphs may have the same quotient graph. However, if we assume that the underlying action is free, then a gain labeling makes the relation one-to-one. To see this, we arbitrary choose a vertex  $v$  as a representative vertex from each vertex orbit. Then, each orbit is written by  $\Gamma v = \{gv \mid g \in \Gamma\}$ . If the action is free, an edge orbit connecting  $\Gamma u$  and  $\Gamma v$  in  $H/\Gamma$  can be written by  $\{\{gu, ghv\} \mid g \in \Gamma\}$  for a unique  $h \in \Gamma$ . We then orient the edge orbit from  $\Gamma u$  to  $\Gamma v$  in  $H/\Gamma$  and assign to it the gain  $h$ . In this way, we obtain *the quotient  $\Gamma$ -gain graph*, denoted  $(H/\Gamma, \psi)$ .

Conversely, let  $(G, \psi)$  be a finite  $\Gamma$ -gain graph for a group  $\Gamma$ . We simply denote the pair  $(g, v)$  of  $g \in \Gamma$  and  $v \in V(G)$  by  $gv$ . The *covering graph* (also known as the derived graph) of  $(G, \psi)$  is the simple graph with the vertex set  $\Gamma \times V(G) = \{gv \mid g \in \Gamma, v \in V(G)\}$  and the edge set  $\{\{gu, g\psi(e)v\} \mid e = (u, v) \in E(G), g \in \Gamma\}$ .

Clearly,  $\Gamma$  freely acts on the covering graph with the action  $\theta$  defined by  $\theta(g) : v \mapsto gv$  for  $g \in \Gamma$ , under which the quotient graph comes back to  $(G, \psi)$ . In this way, there is a one-to-one correspondence between  $\Gamma$ -gain graphs and  $\Gamma$ -symmetric graphs with free actions. For more properties of covering graphs, see e.g., [1, 13].

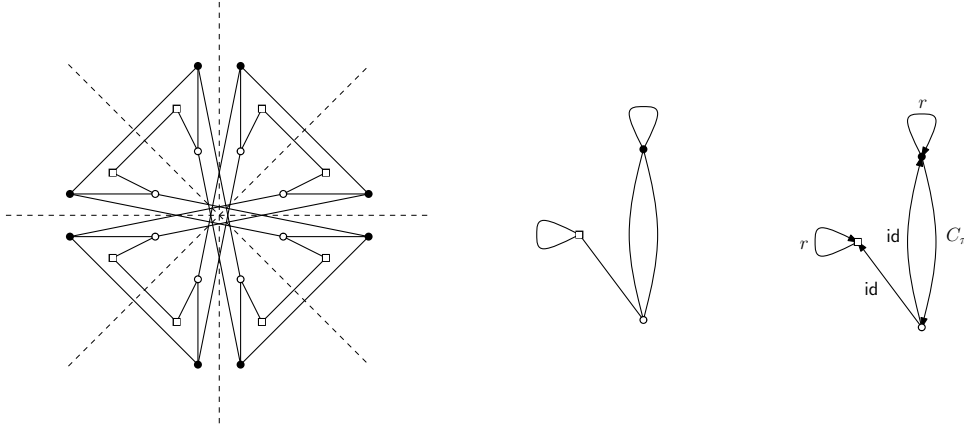


Figure 3: A  $\mathcal{D}_4$ -symmetric graph, the quotient graph, and the quotient  $\mathcal{D}_4$ -gain graph, for dihedral group  $\mathcal{D}_4$  of order 4.

## 6.2 Finite Symmetric frameworks

A  $d$ -dimensional framework (or, simply, a framework) is a pair  $(H, p)$  of a simple undirected graph  $H$  and a mapping  $p : V(H) \rightarrow \mathbb{R}^d$ , called a *point-configuration*, which may be regarded as a straight-line realization of  $G$  in  $\mathbb{R}^d$ .

In this paper, we are interested in symmetrically embedded symmetric graphs in the Euclidean space. To define it formally, we shall introduce some notations. Let  $H$  be a  $\Gamma$ -symmetric graph, where  $\Gamma$  freely acts on  $H$ , and suppose that  $\Gamma$  also acts on  $\mathbb{R}^d$ . A function  $f : V(H) \rightarrow \mathbb{R}^d$  is called  $\Gamma$ -symmetric if

$$\gamma f(v) = f(\gamma v) \quad \forall \gamma \in \Gamma \text{ and } \forall v \in V(H). \quad (32)$$

The pair  $(H, p)$  is said to be a  $\Gamma$ -symmetric framework if  $H$  and  $p$  are  $\Gamma$ -symmetric.

It is convenient to fix a representative vertex  $v$  of each vertex orbit  $\Gamma v$ , and define a *quotient* of  $f$  to be  $f/\Gamma : V/\Gamma \rightarrow \mathbb{R}^d$ , where there is a one-to-one correspondence between  $f$  and  $f/\Gamma$  through  $f(v) = f/\Gamma(\Gamma v)$ .

In the subsequent discussion, we shall only consider finite frameworks, where  $H$  is a finite graph. (In §10, we will discuss infinite frameworks with crystallographic symmetry.) Namely, we shall restrict our attention to *discrete point groups*  $\mathcal{P}$ , which are finite discrete subgroups of the *orthogonal group*  $\mathcal{O}(\mathbb{R}^d)$ , i.e., the set of  $d \times d$  orthogonal matrices.

For a discrete point group  $\Gamma$ , let  $\mathbb{Q}_\Gamma$  be the field generated by  $\mathbb{Q}$  and the entries of matrices contained in  $\Gamma$ . For a  $\Gamma$ -gain graph  $(G, \psi)$ , a mapping  $f : V(G) \rightarrow \mathbb{R}^d$  is said to be  $\Gamma$ -generic if the set of coordinates of the image of  $f$  is algebraically independent over  $\mathbb{Q}_\Gamma$ . Also, for a  $\Gamma$ -symmetric graph  $H$ , a  $\Gamma$ -symmetric function  $f : V(H) \rightarrow \mathbb{R}^d$  is said to be  $\Gamma$ -generic if  $f/\Gamma$  is  $\Gamma$ -generic.

## 6.3 Symmetric parallel redrawing problem

### 6.3.1 Parallel redrawings

Let  $(H, p)$  be a finite  $d$ -dimensional framework. We shall consider  $(H, p)$  as a *drawing* of the graph  $H$  in  $\mathbb{R}^d$  with straight-line edges.

A framework  $(H, q)$  is called a *parallel redrawing* of  $(H, p)$  if  $q(i) - q(j)$  is parallel to  $p(i) - p(j)$  for all  $\{i, j\} \in E(H)$ . No matter how the underlying graph is dense, any drawing admits redrawings, since a translation of  $(H, p)$  or a dilation of  $(H, p)$  is always a redrawing. A drawing  $(H, p)$  is said to be *robust* if any redrawing of  $(H, p)$  is a consequence of translations and dilation of  $(H, p)$ . In the parallel redrawing problem, we are asked whether  $(H, p)$  is robust or not.

In the context of rigidity theory, this concept is known as the *direction-rigidity* of  $d$ -dimensional bar-joint frameworks  $(H, \mathbf{p})$ , where we are interested in direction-constraint, rather than conventional length-constraint (which we will discuss in the next subsection), or the mixture of length and direction constraints (see, e.g., [39, 15, 34]).

Let us take a look at the formal definition. We define a *relocation* of  $(H, p)$  by  $m : V(H) \rightarrow \mathbb{R}^d$  such that

$$m(i) - m(j) \text{ is parallel to } p(i) - p(j) \quad \forall \{i, j\} \in E(H). \quad (33)$$

For  $t \in \mathbb{R}^d$ , let us define  $m_t : V(H) \rightarrow \mathbb{R}^d$  by  $m_t(v) = t$  for  $v \in V(H)$ . Then,  $m_t$  is a relocation of  $(H, p)$ , known as a *translation*. On the other hand, let  $m_{\text{di}}(v) = p(v)$  for

$v \in V(H)$ . Then,  $m_{\text{di}}$  is also a relocation of  $(H, p)$ , known as a *dilation*. In general, a relocation is called *trivial* if it can be written as a linear combination of  $m_{\text{di}}$  and  $m_t$  for  $t \in \mathbb{R}^d$ , and  $(H, p)$  is called *robust* if all possible relocations are trivial. The set of trivial relocations of  $(G, p)$  forms a linear subspace of  $(\mathbb{R}^d)^V$ , denoted by  $\text{tri}(H, p)$ , which has dimension  $d + 1$  unless  $\{p(v) \mid v \in V(H)\}$  is a point.

In [39], Whiteley showed a combinatorial characterization of robust frameworks on generic point-configurations, as a corollary of a combinatorial characterization of reconstructivity of pictures appeared in scene analysis (see, e.g., [38, 39] or see also [35]). The goal of this section is to extend this result to the *symmetric redrawing problem* of symmetric frameworks.

Let  $\Gamma$  be a discrete point group in  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ , and suppose that  $(H, p)$  is  $\Gamma$ -symmetric. It is then natural to ask whether there is a redrawing preserving the symmetry. Namely, we shall take into account only  $\Gamma$ -*symmetric relocations*. It is straightforward from (32)(33) to see the following:

**Proposition 6.1.** *Let  $(H, p)$  be a  $\Gamma$ -symmetric framework, and let  $m : V \rightarrow \mathbb{R}^d$  be a  $\Gamma$ -symmetric relocation of  $(H, p)$ . Then,  $(H, p + m)$  is  $\Gamma$ -symmetric and a parallel-redrawing of  $(H, p)$ . Conversely, if  $(H, q)$  is  $\Gamma$ -symmetric and a parallel-redrawing of  $(H, p)$ , then  $q - p$  is a  $\Gamma$ -symmetric relocation of  $(H, p)$ .*

We thus say that  $(H, p)$  is *symmetrically robust* if any  $\Gamma$ -symmetric relocation of  $(H, p)$  is trivial. The set of  $\Gamma$ -symmetric trivial relocations forms a linear subspace of  $\text{tri}(H, p)$ , denoted by  $\text{tri}_\Gamma(H, p)$ .

Recall that the space of relocations of  $(H, p)$  has dimension at least  $d + 1$ , as  $\dim(\text{tri}(H, p)) = d + 1$ . However, not every translation  $m_t$  is  $\Gamma$ -symmetric;  $m_t$  is  $\Gamma$ -symmetric if and only if  $\gamma t = t$  for all  $\gamma \in \Gamma$ , or equivalently  $t \in \bigcap_{\gamma \in \Gamma} \ker(\gamma - I_d)$ . Thus, if  $\{p(v) \mid v \in V(H)\}$  does not lie on a point,  $(H, p)$  is symmetrically robust if and only if the dimension of the space of  $\Gamma$ -symmetric relocations is equal to

$$\text{tri}_\Gamma(H, p) = 1 + \dim\left(\bigcap_{\gamma \in \Gamma} \ker(I_d - \gamma)\right). \quad (34)$$

Note that the dilation is always  $\Gamma$ -symmetric, which is indeed crucial in the proof of our main claim.

### 6.3.2 Symmetric parallel redrawing polymatroids

For a relocation of  $\Gamma$ -symmetric framework  $(H, p)$ , the system (33) is apparently redundant due to the  $\Gamma$ -symmetry of  $p$  and  $m$ , and the redundancy can be eliminated by using the quotient  $\Gamma$ -gain graph  $(H/\Gamma, \psi)$ .

To see this, let us take a representative vertex  $v \in V(H)$  from each vertex orbit  $\Gamma v \in V(H/\Gamma)$ , as mentioned in § 6.2. Then, there is a natural one-to-one correspondence between  $p$  and its quotient  $p/\Gamma$  (resp.,  $m$  and  $m/\Gamma$ ). Recall also that each edge orbit connecting from  $\Gamma i$  to  $\Gamma j$  is written by  $\Gamma e = \{\{\gamma i, \gamma \psi_e j\} : \gamma \in \Gamma\}$ , where  $\psi_e$  is the gain of  $\Gamma e$  in the quotient gain graph. Hence, (33) is written by

$$\langle m(\gamma i) - m(\gamma \psi_e j), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p(\gamma i) - p(\gamma \psi_e j), \alpha \rangle = 0$$

for each edge  $\{\gamma i, \gamma \psi_e j\}$  in an edge orbit  $\Gamma e$ . Since  $\psi_e \in \mathcal{O}(\mathbb{R}^d)$ , these are equivalent to one condition,

$$\langle m(i) - m(\psi_e j), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p(i) - p(\psi_e j), \alpha \rangle = 0,$$

which is further converted to

$$\langle m/\Gamma(\Gamma i) - \psi_e m/\Gamma(\Gamma j), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p/\Gamma(\Gamma i) - \psi_e p/\Gamma(\Gamma j), \alpha \rangle = 0.$$

Thus, the problem can be considered in a general  $\Gamma$ -gain graph  $(G, \psi)$ , and it suffices to analyze the space of  $m \in (\mathbb{R}^d)^{V(G)}$  satisfying

$$\langle m(i), -\psi_e m(j), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p(i) - \psi_e p(j), \alpha \rangle = 0 \quad (35)$$

for every  $e = (i, j) \in E(G)$  with  $\psi(e) = \psi_e$ . To do it, we associate a  $(d-1)$ -dimensional linear subspace  $P_{e,\psi}(p)$  with each edge orbit  $e = (i, j) \in E(G)$ , defined by

$$P_{e,\psi}(p) = A_{e,\psi} \cap \{x \in (\mathbb{R}^d)^{V(G)} \mid \langle p(i) - \psi_e p(j), x(i) \rangle = 0\} \quad (36)$$

where  $A_{e,\psi}$  is, as defined in (27)(28),

$$A_{e,\psi} = \left\{ x \in (\mathbb{R}^d)^{V(G)} \mid \begin{array}{l} x(i) + \psi_e x(j) = 0, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\} \quad (37)$$

or

$$A_{e,\psi} = \left\{ x \in (\mathbb{R}^d)^{V(G)} \mid \exists \alpha \in \mathbb{R}^d: \begin{array}{l} x(i) = (I_d - \psi_e)\alpha, \\ x(V \setminus \{i\}) = 0 \end{array} \right\} \quad (38)$$

depending on whether  $e$  is a non-loop or a loop, respectively. Observe then that  $m \in (\mathbb{R}^d)^{V(G)}$  satisfies (35) if and only if  $m$  is in the orthogonal complement of  $\text{span}\{P_{e,\psi}(p) \mid e \in E(G)\}$ , because, for any  $x \in P_{e,\psi}(p)$ , we have

$$\begin{aligned} \langle m, x \rangle &= \langle m(i), x(i) \rangle + \langle m(j), x(j) \rangle \\ &= \langle m(i), x(i) \rangle - \langle m(j), \psi_e^{-1} x(i) \rangle \\ &= \langle m(i) - \psi_e m(j), x(i) \rangle \end{aligned}$$

with  $\langle p(i) - \psi_e p(j), x(i) \rangle = 0$ . In total, we proved the following:

**Theorem 6.2.** *Let  $(H, p)$  be a  $\Gamma$ -symmetric framework, and  $(H/\Gamma, \psi)$  the quotient  $\Gamma$ -gain graph of  $H$ . Then,  $(H, p)$  is a symmetrically robust drawing if and only if*

$$\dim(\text{span}\{P_{e,\psi}(p/\Gamma) \mid e \in E(H/\Gamma)\}) = d|V/\Gamma| - 1 - \dim\left(\bigcap_{\gamma \in \Gamma} (\ker(\gamma - I_d))\right).$$

### 6.3.3 Combinatorial characterization

By Theorem 6.2, it now suffices to analyze the polymatroid of  $\Gamma$ -gain graphs  $(G, \psi)$  with the linear representation  $e \mapsto P_{e,\psi}(p)$  for  $p : V(G) \rightarrow \mathbb{R}^d$ , which we call the  $\Gamma$ -symmetric parallel redrawing polymatroid of  $(G, \psi)$ . The following theorem provides a combinatorial characterization of this polymatroid.

**Theorem 6.3.** *Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph for a discrete point group  $\Gamma$  and  $p : V \rightarrow \mathbb{R}^d$  be a  $\Gamma$ -generic mapping. Define  $h_\rho : 2^E \rightarrow \mathbb{Z}$  by*

$$h_\rho(F) = f_\rho(F) - 1 \quad (F \subseteq E), \quad (39)$$

where  $f_\rho$  is as defined in (30) with the natural representation  $\rho : \gamma \in \Gamma \mapsto \gamma \in GL(\mathbb{R}^d)$ . Then,

$$\dim(\text{span}\{P_{e,\psi}(p) \mid e \in E\}) = \hat{h}_\rho(E).$$

In other words, for almost all  $p$ , the  $\Gamma$ -symmetric parallel redrawing matroid is equal to the polymatroid induced by  $h_\rho$ .

*Proof.* The proof idea is exactly the same as the alternative proof of Laman's theorem by Lovász and Yemini[19].

Applying Theorem 5.3 with  $f_\rho$ , we have that the polymatroid  $\mathbf{P}(f_\rho) = (E, f_\rho)$  is equal to the linear polymatroid  $\mathbf{DP}_\rho(G, \psi)$  with the linear representation  $e \mapsto A_{e,\psi}$  (given in (37)(38)).

We define a hyperplane  $H$  of  $(\mathbb{R}^d)^V$  (i.e., a  $(d|V| - 1)$ -dimensional subspace) by

$$H = \{x \in (\mathbb{R}^d)^V \mid \langle p, x \rangle = 0\}$$

where  $p \in (\mathbb{R}^d)^V$  is  $\Gamma$ -generic as defined in the statement. Then, observe that  $P_{e,\psi}(p) = A_{e,\psi} \cap H$  for every  $e \in E$ ; indeed, for any  $e = (i, j) \in E$  and any  $x \in A_{e,\psi}$ ,

$$\begin{aligned} \langle p, x \rangle &= \langle p(i), x(i) \rangle + \langle p(j), x(j) \rangle \\ &= \langle p(i), x(i) \rangle + \langle p(j), -\psi_e^{-1}x(i) \rangle \\ &= \langle p(i) - \psi_e p(j), x(i) \rangle, \end{aligned}$$

implying  $x \in H$  if and only if  $\langle p(i) - \psi_e p(j), x(i) \rangle = 0$  for every  $e = (i, j)$ . Therefore, as  $p$  is  $\Gamma$ -generic, we conclude that the  $\Gamma$ -symmetric parallel redrawing polymatroid of  $(G, \psi)$  is obtained from  $\mathbf{DP}_\rho(G, \psi)$  by a Dilworth truncation, given in §3.4.2. By Theorem 3.2 and Theorem 5.3, we finally obtain

$$\begin{aligned} &\dim(\text{span}\{P_{e,\psi}(p) \mid e \in E\}) \\ &= \min\left\{\sum_i (\dim(\text{span}\{A_{e,\psi} \mid e \in E_i\}) - 1) \mid \text{a partition}\{E_1, \dots, E_k\} \text{ of } E\right\} \quad (\text{by Theorem 3.2}) \\ &= \min\left\{\sum_i (f_\rho(E_i) - 1) \mid \text{a partition}\{E_1, \dots, E_k\} \text{ of } E\right\} \quad (\text{by Theorem 5.3}) \\ &= \min\left\{\sum_i h_\rho(E_i) \mid \text{a partition}\{E_1, \dots, E_k\} \text{ of } E\right\} \quad (\text{by (39)}) \\ &= \hat{h}_\rho(E). \quad (\text{by (3)}). \end{aligned}$$

□

The following result extends the result of Whiteley [39] to symmetric drawings, which directly follows from Theorem 6.2 and Theorem 6.3.

**Corollary 6.4.** *Let  $H$  be a  $\Gamma$ -symmetric graph for a discrete point group  $\Gamma$  and  $(H/\Gamma, \psi)$  be the quotient  $\Gamma$ -gain graph. For almost all  $\Gamma$ -symmetric  $p : V(H) \rightarrow \mathbb{R}^d$ ,  $(H, p)$  is symmetrically robust if and only if the graph obtained from  $H/\Gamma$  by replacing each edge  $e \in E(H/\Gamma)$  by  $d-1$  parallel copies contains an edge subset  $I$  satisfying the following counting conditions:*

- $|I| = d|V| - 1 - \dim(\bigcap_{\gamma \in \Gamma} \ker(\gamma - I_d))$ ;
- $|F| \leq d|V(F)| - dc(F) - 1 + \sum_{X \in C(F)} \dim(\text{span}\{\text{im}(\gamma - I_d) \mid \gamma \in \langle X \rangle\})$  for any nonempty  $F \subseteq I$ .

## 6.4 Symmetric rigidity of symmetric frameworks

We then move to another application of Theorem 5.3, the infinitesimal rigidity of symmetric frameworks. The description of §6.4.1 and §6.4.2 is taken from [16], which contains a more detailed explanation on this topic.

### 6.4.1 Symmetric infinitesimal rigidity

The *infinitesimal rigidity* concerns with the dimension of the space of infinitesimal motions. An *infinitesimal motion* of a framework  $(H, p)$  is defined as an assignment  $m : V(H) \rightarrow \mathbb{R}^d$  such that

$$\langle m(i) - m(j), p(i) - p(j) \rangle = 0 \quad \forall \{i, j\} \in E(H). \quad (40)$$

The set of infinitesimal motions forms a linear space, denoted  $L(H, p)$ .

In general, for a set  $P \subseteq \mathbb{R}^d$  of points, an *infinitesimal isometry* of  $P$  is defined by  $m : P \rightarrow \mathbb{R}^d$  such that

$$\langle m(x) - m(y), x - y \rangle = 0 \quad \forall x, y \in P.$$

The set of infinitesimal isometries forms a linear space, denoted by  $\text{iso}(P)$ . Notice that, for a skew-symmetric matrix  $S$  and  $t \in \mathbb{R}^d$ , a mapping  $m : P \rightarrow \mathbb{R}^d$  defined by

$$m(x) = Sx + t \quad (x \in P)$$

is an infinitesimal isometry of  $P$ . Indeed, it is well-known that any infinitesimal isometry can be described in this form, and

$$\dim \text{iso}(P) = d(k+1) - \binom{k+1}{2}, \quad (41)$$

where  $k$  denotes the affine dimension of  $P$ . For example, for  $d = 2$ , an infinitesimal isometry is a linear combination of *translations* and *infinitesimal rotations*.

An infinitesimal motion  $m : V(H) \rightarrow \mathbb{R}^d$  of a framework  $(H, p)$  is said to be *trivial* if  $m$  can be expressed by

$$m(v) = Sp(v) + t \quad (v \in V(H)) \quad (42)$$

for some skew-symmetric matrix  $S$  and  $t \in \mathbb{R}^d$ . The set of all trivial motions forms a linear subspace of  $L(H, p)$ , denoted by  $\text{tri}(H, p)$ . By definition,  $\text{tri}(H, p)$  is isomorphic to  $\text{iso}(\{p(v) \mid v \in V(H)\})$ , and hence (41) gives the exact dimension of  $\text{tri}(H, p)$ .  $(H, p)$  is called *infinitesimally rigid* if  $L(H, p) = \text{tri}(H, p)$ .

As in the parallel redrawing problem, we are interested in  $\Gamma$ -symmetric infinitesimal motions of symmetric frameworks. For a discrete point group  $\Gamma$ , a  $\Gamma$ -symmetric framework  $(H, p)$  is said to be *symmetrically infinitesimally rigid* if any  $\Gamma$ -symmetric infinitesimal motion of  $(H, p)$  is trivial. We should remark that, as in the case of the parallel redrawing problem, not every trivial infinitesimal motion is  $\Gamma$ -symmetric.

The following result of Schulze [32] motivates us to look at  $\Gamma$ -symmetric infinitesimal rigidity. (The precise definition of some terminologies are omitted here.)



**Theorem 6.5** (Schulze [32]). *Let  $\Gamma$  be a discrete point group, and  $H$  be a  $\Gamma$ -symmetric graph. Then, for almost all  $\Gamma$ -symmetric point-configurations  $p$ ,  $(H, p)$  has a nontrivial continuous motion that preserves the  $\Gamma$ -symmetry if and only if  $(H, p)$  is not symmetrically infinitesimally rigid.*

#### 6.4.2 Orbit rigidity matrix

Let  $(H, p)$  be a  $\Gamma$ -symmetric framework. Due to  $\Gamma$ -symmetry, the system (40) of linear equations (with respect to  $m$ ) is redundant. Schulze and Whiteley [33] pointed that the system can be reduced to  $|E(H)/\Gamma|$  linear equations.

Indeed, by the same manner as §6.3.2, we can reduce that  $m : V(H) \rightarrow \mathbb{R}^d$  is a  $\Gamma$ -symmetric infinitesimal motion of  $(H, p)$  if and only if

$$\langle m/\Gamma(\Gamma i), p/\Gamma(\Gamma i) - \psi_e \cdot p/\Gamma(\Gamma j) \rangle + \langle m/\Gamma(\Gamma j), p/\Gamma(\Gamma j) - \psi_e^{-1} \cdot p/\Gamma(\Gamma i) \rangle = 0 \quad (43)$$

for every oriented edge orbit  $\Gamma e$  in the quotient gain graph  $(H/\Gamma, \psi)$ . By regarding (43) as a system of linear equations of  $m/\Gamma$ , the corresponding  $|E(H)/\Gamma| \times d|V(H)/\Gamma|$ -matrix is called the *orbit rigidity matrix* by Schulze and Whiteley [33].

In general, for a  $\Gamma$ -gain graph  $(G, \psi)$  and  $p : V(G) \rightarrow \mathbb{R}^d$ , we shall consider the system of linear equations on  $m \in (\mathbb{R}^d)^{V(G)}$  defined by

$$\langle m(i), p(i) - \psi_e p(j) \rangle + \langle m(j), p(j) - \psi_e^{-1} p(i) \rangle = 0 \quad \forall e = (i, j) \in E(G). \quad (44)$$

To analyze the solution space of (44), we then associate a 1-dimensional linear space  $R_{e, \psi}$  with each  $e = (i, j) \in E(G)$ ,

$$R_{e, \psi}(p) = \left\{ x \in (\mathbb{R}^d)^{V(G)} \mid \exists t \in \mathbb{R} : \begin{array}{l} x(i) = t(p(i) - \psi_e p(j)), \\ x(j) = t(p(j) - \psi_e^{-1} p(i)), \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\} \quad (45)$$

if  $e$  is a non-loop edge, and

$$R_{e, \psi}(p) = \left\{ x \in (\mathbb{R}^d)^{V(G)} \mid \exists t \in \mathbb{R} : \begin{array}{l} x(i) = t(2I_d - \psi_e - \psi_e^{-1})p(i), \\ x(V \setminus \{i\}) = 0 \end{array} \right\} \quad (46)$$

if  $e$  is a loop attached to  $i$ . Observe then that  $m$  satisfies (44) if and only if  $m$  is in the orthogonal complement of  $\text{span}\{R_{e, \psi}(p) \mid e \in E\}$ . This implies the following:

**Proposition 6.6** (Schulze and Whiteley [33]). *Let  $(H, p)$  be a  $\Gamma$ -symmetric framework and  $(H/\Gamma, \psi)$  be the quotient  $\Gamma$ -gain graph. Then, the dimension of the space of  $\Gamma$ -symmetric infinitesimal motions of  $(H, p)$  is equal to*

$$d|V(H)/\Gamma| - \dim(\text{span}\{R_{e, \psi}(p/\Gamma) \mid e \in E(H/\Gamma)\}).$$

The detailed description and examples can be found in [32, 33, 28]. A combinatorial necessity condition of  $\Gamma$ -symmetric infinitesimal rigidity can be found in [16].

### 6.4.3 Combinatorial characterization

The following theorem provides a combinatorial characterization of the linear matroid induced on  $\{R_{e,\psi} \mid e \in E(G)\}$  for the spacial case of  $d = 2$  and rotation groups  $\mathcal{C}_k$ .

**Theorem 6.7.** *Let  $\mathcal{C}_k$  be the group of  $k$ -fold rotations around the origin in the plane. Let  $(G = (V, E), \psi)$  be a  $\mathcal{C}_k$ -gain graph and  $p : V \rightarrow \mathbb{R}^2$  be a  $\mathcal{C}_k$ -generic mapping. Then,*

$$\dim(\text{span}\{R_{e,\psi}(p) \mid e \in E\}) = \hat{h}_\rho(E),$$

where  $h_\rho$  is as defined in (39) with  $\Gamma = \mathcal{C}_k$ .

*Proof.* The proof technique is exactly the same as the proof of Theorem 6.3.

Let  $C_{\pi/2}$  be the matrix of size  $2 \times 2$ , representing the 4-fold rotation around the origin in the Euclidean plane. We define a hyperplane  $H'$  of  $(\mathbb{R}^2)^V$  by

$$H' = \{x \in (\mathbb{R}^2)^V \mid \sum_{v \in V} \langle C_{\pi/2} p(v), x(v) \rangle = 0\}$$

where  $p \in (\mathbb{R}^2)^V$  is  $\mathcal{C}_k$ -generic as defined in the statement.

Let  $A_{e,\psi}$  be the 2-dimensional linear subspace defined in (37)(38) with  $d = 2$ . Then, observe that, for each  $e = (i, j) \in E$  and for any  $x \in A_{e,\psi}$ ,

$$\begin{aligned} \sum_{v \in V} \langle C_{\pi/2} p(v), x(v) \rangle &= \langle C_{\pi/2} p(i), x(i) \rangle + \langle C_{\pi/2} p(j), x(j) \rangle \\ &= \langle C_{\pi/2} p(i), x(i) \rangle + \langle C_{\pi/2} p(j), -\psi_e^{-1} x(i) \rangle \\ &= \langle C_{\pi/2} (p(i) - \psi_e p(j)), x(i) \rangle \end{aligned}$$

where we used the fact that  $C_{\pi/2}$  commutes with any element of  $\mathcal{C}_k$ . This implies that  $x \in H'$  if and only if  $x(i) \in \text{span}\{p(i) - \psi_e p(j)\}$  for every  $e = (i, j) \in E$ . In other words,  $R_{e,\psi}(p) = A_{e,\psi} \cap H'$ .

Since  $p$  is  $\mathcal{C}_k$ -generic, we conclude that the linear matroid induced on  $\{R_{e,\psi}(p) \mid e \in E\}$  is obtained from  $\mathbf{DP}_\rho(G, \psi)$  by a Dilworth truncation. Since  $\mathbf{DP}_\rho(G, \psi) = \mathbf{P}(f_\rho)$  by Theorem 5.3, Theorem 3.2 implies the statement. (For a concrete description, see the last paragraph of the proof of Theorem 6.3.)  $\square$

Combining Proposition 6.6 and Theorem 6.7, we conclude that the row matroid of an orbit rigidity matrix is the matroid induced by  $h_\rho$ , if  $d = 2$  and the underlying symmetry is a group of rotations.

## 7 Matroids Induced by Group Actions on Exterior Product Spaces

### 7.1 Restriction to decomposable $k$ -vectors

For another application to rigidity theory, we shall investigate the case when a group is represented in the exterior product of vector spaces. In the subsequent discussion of this section, the underlying group  $\Gamma$  is equipped with a linear representation  $\rho : \Gamma \rightarrow GL(\mathbb{F}^d)$

over a field  $\mathbb{F}$ . As before, we denote by  $\mathbb{K}$  the field obtained from  $\mathbb{F}$  by transcendental extensions.

Let  $\bigwedge^k \mathbb{F}^d$  be the  $k$ -th exterior power of  $\mathbb{F}^d$ . Recall that  $\bigwedge^k \mathbb{F}^d$  is a  $\binom{d}{k}$ -dimensional linear space, and so each entry of an element of  $\bigwedge^k \mathbb{F}^d$  can be naturally indexed by a  $k$ -tuple  $(i_1, \dots, i_k)$  with  $1 \leq i_1 < \dots < i_k \leq d$ . An element of  $\bigwedge^k \mathbb{F}^d$  is called a  $k$ -vector, and a  $k$ -vector is said to be *decomposable* if it can be written of the form  $v_1 \wedge \dots \wedge v_k$ . Recall that  $\bigwedge^k \mathbb{F}^d$  is spanned by decomposable  $k$ -vectors.

Let us consider a natural action of  $\Gamma$  on  $\bigwedge^k \mathbb{F}^d$ , where  $\gamma \in \Gamma$  acts on a decomposable element  $v_1 \wedge \dots \wedge v_k$  by  $\rho(\gamma)v_1 \wedge \dots \wedge \rho(\gamma)v_k$  and extends linearly to the other elements. This leads to a linear mapping from  $\bigwedge^k \mathbb{F}^d$  to  $\bigwedge^k \mathbb{F}^d$ . It is known that this action is a well-defined group representation of  $\Gamma$  over  $GL(\bigwedge^k \mathbb{F}^d)$ . In other words, there is a unique representation  $\rho^{(k)} : \Gamma \rightarrow GL(\bigwedge^k \mathbb{F}^d)$  such that  $\rho^{(k)}(\gamma)(v_1 \wedge \dots \wedge v_k) = \rho(\gamma)v_1 \wedge \dots \wedge \rho(\gamma)v_k$  for each  $\gamma \in \Gamma$  and each  $v_1 \wedge \dots \wedge v_k$  (see e.g., [14, Chapter 7]).

Note that each  $\rho^{(k)}(\gamma)$  is a matrix of size  $\binom{d}{k} \times \binom{d}{k}$ . To see a specific expression of the entries, let us a matrix  $A = \rho(\gamma)$ . For  $1 \leq i_1 < \dots < i_k \leq d$  and  $1 \leq j_1 < \dots < j_k \leq d$ , let  $A_{i_1 \dots i_k}^{j_1 \dots j_k}$  be the submatrix of  $A$  induced by the  $i_1$ -th,  $\dots$ ,  $i_k$ -th rows and the  $j_1$ -th,  $\dots$ ,  $j_k$ -th columns. If we index each column and each row of  $A^{(k)}$  by a  $k$ -tuple  $(i_1, \dots, i_k)$  according to the index ordering of elements in  $\bigwedge^k \mathbb{F}^d$ , we have

$$A^{(k)}[(j_1, \dots, j_k), (i_1, \dots, i_k)] = \det A_{i_1 \dots i_k}^{j_1 \dots j_k},$$

where  $A^{(k)}[(j_1, \dots, j_k), (i_1, \dots, i_k)]$  denotes the entry of the intersection of the  $(j_1, \dots, j_k)$ -th row and  $(i_1, \dots, i_k)$ -th column.

Using the representation  $\rho^{(k)}$ , we consider a special case of matroids given in the previous section. Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph with a gain function  $\psi : e \mapsto \psi_e$ . For each non-loop edge  $e = (i, j) \in E$ , let us assign  $x_{e, \psi}^{(k)} \in (\bigwedge^k \mathbb{K}^d)^V$  as follows:

$$x_{e, \psi}^{(k)}(v) = \begin{cases} -\rho^{(k)}(\psi_e)\alpha_e & \text{if } v = i \\ \alpha_e & \text{if } v = j \\ 0 & \text{otherwise} \end{cases} \quad (47)$$

$$x_{e, \psi}^{(k)}(v) = \begin{cases} (\rho^{(k)}(\psi_e) - I_{\binom{d}{k}})\alpha_e & \text{if } v = i \\ 0 & \text{otherwise} \end{cases} \quad (48)$$

depending on whether  $e$  is a non-loop edge or a loop, where  $\alpha_e = (\alpha_e^1, \dots, \alpha_e^{\binom{d}{k}})^\top \in \bigwedge^k \mathbb{K}^d$  such that  $\{\alpha_e^i \mid 1 \leq i \leq \binom{d}{k}, e \in E\}$  is algebraically independent over  $\mathbb{F}$ . Then, by Corollary 5.5,  $\{x_{e, \psi}^{(k)} \mid e \in E\}$  is linearly independent if and only if  $|F| \leq f_{\rho^{(k)}}(F)$  for any  $F \subseteq E$ , where

$$f_{\rho^{(k)}}(F) = \binom{d}{k}|V(F)| - \binom{d}{k}c(F) + \sum_{X \in C(F)} d_{\rho^{(k)}}\langle X \rangle, \quad (49)$$

$$d_{\rho^{(k)}}(X) = \dim(\text{span}\{\text{im}(\rho^{(k)}(\gamma) - I_{\binom{d}{k}}) \mid \gamma \in X\}) \quad (X \subseteq \Gamma). \quad (50)$$

Note that, for any  $\gamma \in \Gamma$ , we have

$$\text{im}(\rho^{(k)}(\gamma) - I_{\binom{d}{k}}) = \text{span}\{\rho(\gamma)p_1 \wedge \dots \wedge \rho(\gamma)p_k - p_1 \wedge \dots \wedge p_k \mid p_1, \dots, p_k \in \mathbb{F}^d\},$$

and hence  $d_{\rho^{(k)}}$  can be rewritten in terms of decomposable  $k$ -vectors by putting this into (50). It is hence natural to ask whether the matroid  $\mathbf{M}(f_{\rho^{(k)}})$  has a linear representation in terms of decomposable  $k$ -vectors, that is, a representation given by

$$\hat{x}_{e,\psi}^{(k)}(v) = \begin{cases} -\rho(\psi_e)p_{e,1} \wedge \cdots \wedge \rho(\psi_e)p_{e,k} & \text{if } v = i \\ p_{e,1} \wedge \cdots \wedge p_{e,k} & \text{if } v = j \\ 0 & \text{otherwise} \end{cases} \quad (51)$$

and

$$\hat{x}_{e,\psi}^{(k)}(v) = \begin{cases} \rho(\psi_e)p_{e,1} \wedge \cdots \wedge \rho(\psi_e)p_{e,k} - p_{e,1} \wedge \cdots \wedge p_{e,k} & \text{if } v = i \\ 0 & \text{otherwise} \end{cases} \quad (52)$$

for some  $p_{e,1}, \dots, p_{e,k} \in \mathbb{K}^d$ . The next theorem asserts that (51)(52) indeed define a linear representation of  $\mathbf{M}(f_{\rho^{(k)}})$ .

**Theorem 7.1.** *The linear matroid induced on  $\{\hat{x}_{e,\psi}^{(k)} \mid e \in E\}$  is equal to the matroid  $\mathbf{M}(f_{\rho^{(k)}})$ , for some  $\{p_{e,i} \in \mathbb{K}^d \mid e \in E, 1 \leq i \leq k\}$ .*

*Proof.* Let

$$A_{e,\psi}^{(k)} = \left\{ x \in (\wedge^k \mathbb{K}^d)^V \mid \begin{array}{l} x(i) + \rho^{(k)}(\psi_e)x(j) = 0, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\}$$

for a non-loop edge  $e \in E$ , and

$$A_{e,\psi}^{(k)} = \left\{ x \in (\wedge^k \mathbb{K}^d)^V \mid \exists \alpha \in (\wedge^k \mathbb{K}^d)^V : \begin{array}{l} x(i) = (I_d - \rho^{(k)}(\psi_e))\alpha, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\}$$

for a loop  $e \in E$ .

Let  $\text{Gr}(d, k)$  be the set of all decomposable  $k$ -vectors in  $\wedge^k \mathbb{K}^d$ .  $\text{Gr}(d, k)$  is known as the *Grassmannian* in the literature and is an irreducible rational variety spanning  $\wedge^k \mathbb{K}^d$ . We shall define a subset  $\hat{A}_{e,\psi}^{(k)}$  of  $A_{e,\psi}^{(k)}$  by

$$\hat{A}_{e,\psi}^{(k)} = A_{e,\psi}^{(k)} \cap \{x \in (\wedge^k \mathbb{K}^d)^V \mid x(i) \in \text{Gr}(d, k)\}.$$

for a non-loop edge  $e = (i, j)$ , and

$$\hat{A}_{e,\psi}^{(k)} = \left\{ x \in (\wedge^k \mathbb{K}^d)^V \mid \exists \alpha \in \text{Gr}(d, k) : \begin{array}{l} x(i) = (I_d - \rho^{(k)}(\psi_e))\alpha, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\}$$

for a loop  $e$ .

By Theorem 5.3, we know that  $f_{\rho^{(k)}}(E) = \dim(\text{span}\{A_{e,\psi}^{(k)} \mid e \in E\})$ , and each linear representation of  $\mathbf{M}_{\rho^{(k)}}$  is obtained by taking a representative vector  $x_{e,\psi}^{(k)}$  from each  $A_{e,\psi}^{(k)}$  in generic position. Thus, to show the statement, it is sufficient to show that a representative vector  $x_{e,\psi}^{(k)}$  can be taken from  $\hat{A}_{e,\psi}^{(k)}$  so that  $\{x_{e,\psi}^{(k)} \mid e \in E\}$  is in generic position in the sense of (8).

Suppose that  $E$  contains no loop. Then, each  $\hat{A}_{e,\psi}^{(k)}$  is (linearly) isomorphic to  $\text{Gr}(d, k)$  by a projection to  $x(i)$ . Notice that the condition (8) of genericity is written in terms of linear dependencies. Since  $\text{Gr}(k, d)$  is an irreducible rational variety, the linear isomorphism between  $\hat{A}_{e,\psi}^{(k)}$  and  $\text{Gr}(d, k)$  implies that a representative vertex can be taken from  $\hat{A}_{e,\psi}^{(k)}$  in

generic position. (A more detailed description for a special case can be found in [35, Theorem 3.1], and the exactly same argument can be applied here.)

If  $e$  is a loop, then  $A_{e,\psi}^{(k)}$  and  $\hat{A}_{e,\psi}^{(k)}$  are linearly isomorphic to  $(\rho^{(k)}(\psi_e) - I_d)(\bigwedge^k \mathbb{K}^d)$  and  $(\rho^{(k)}(\psi_e) - I_d)\text{Gr}(d, k)$ , respectively. Since  $\rho^{(k)}(\psi_e) - I_d$  is a linear operator, we can apply the same argument.  $\square$

## 7.2 Symmetric Rigidity of Body-bar Frameworks

As an application, we shall consider matroids from the rigidity of symmetric body-bar frameworks, which is a structure consisting of rigid bodies linked by bars in  $\mathbb{R}^d$ .

Let  $\text{Aff}(\mathbb{R}^d)$  be the group of invertible affine transformations. It is well-known that  $\text{Aff}(\mathbb{R}^d) = GL(\mathbb{R}^d) \ltimes \mathbb{R}^d$ , that is, the semidirect product of  $GL(\mathbb{R}^d)$  and  $\mathbb{R}^d$ , and each element  $(A, t) \in \text{Aff}(\mathbb{R}^d)$  acts on  $\mathbb{R}^d$  by  $(A, t) \cdot q = Aq + t$  for  $q \in \mathbb{R}^d$ . Equivalently, each element  $\gamma \in \Gamma$  is represented by an augmented  $(d+1) \times (d+1)$ -matrix of the form

$$\rho(\gamma) = \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} \in GL(\mathbb{R}^{d+1}). \quad (53)$$

The  $d$ -dimensional Euclidean group  $\mathcal{E}(d)$  is a subgroup of  $\text{Aff}(\mathbb{R}^d)$ , where  $(A, t) \in \text{Aff}(\mathbb{R}^d)$  is in  $\mathcal{E}(d)$  if and only if  $A \in \mathcal{O}(\mathbb{R}^d)$ . A *space group (or crystallographic group)*  $\Gamma$  is a discrete cocompact subgroup of  $\mathcal{E}(d)$ , i.e.,  $\mathbb{R}^d/\Gamma$  is compact. Throughout this subsection,  $\Gamma$  denotes either a space group or a discrete point group, where  $t = 0$  in case of a point group.

Given a  $\Gamma$ -gain graph  $(G, \psi)$  with a crystallographic group  $\Gamma$ , we shall consider the matroids given in the last section with respect to the representation  $\rho$  of  $\Gamma$  given by (53).

We now briefly take a look at how the linear matroid given in the last section arises in the context of rigidity of body-bar frameworks. The following modeling is based on [4]. A body-bar framework can be represented by a triple  $(H, B, q)$ , where

- $H$  is an undirected graph whose vertex is corresponding to a body and whose edge is corresponding to a bar linking the corresponding two bodies;
- $B$  indicates the location of each body by  $B(v) = (A_v, p_v) \in \mathcal{O}(\mathbb{R}^d) \ltimes \mathbb{R}^d$  (i.e., each body is identified with a Cartesian (local) coordinate system) for each  $v \in V$ ;
- $q$  indicates the location of each bar in each local coordinate system as follows; for each  $e \in E(H)$  and an endvertex  $v$  of  $e$ ,  $q(e, v) \in \mathbb{R}^d$  denotes the coordinate of the endpoint the bar associated with  $e$  in the coordinate system of the body  $v$ . Thus, the coordinate in the global system is equal to  $A_v q(e, v) + p_v$ , denoted by  $\tilde{q}(e, v)$ .

$B$  and  $q$  are called *body-configuration* and *bar-configuration*, respectively.

Each bar constraints the distance between the endpoints. Such a length constraint can be written by

$$\langle \tilde{q}(e, i) - \tilde{q}(e, j), \tilde{q}(e, i) - \tilde{q}(e, j) \rangle = \ell_e \quad \forall e = \{i, j\} \in E(H) \quad (54)$$

by some specific bar-length  $\ell_e$ .

We consider a symmetric version of body-bar frameworks, where a body-bar framework  $(H, B, q)$  is  $\Gamma$ -*symmetric* if  $H$  is a  $\Gamma$ -symmetric graph and  $b$  and  $q$  are subject to  $\Gamma$ -symmetry:

for any  $v \in V(H)$ ,  $e \in E(H)$ , and  $\gamma = (A, t) \in \Gamma$ .

$$B(\gamma v) = \rho(\gamma)B(v) = (AA_v, Ap_v + t) \quad (55)$$

$$q(\gamma e, \gamma v) = q(e, v). \quad (56)$$

Indeed, in the global coordinate system, we have  $\tilde{q}(\gamma e, \gamma v) = AA_v q(\gamma e, \gamma v) + Ap_v + t = AA_v q(e, v) + Ap_v + t = \rho(\gamma)\tilde{q}(e, v)$ , and thus the definition implies the  $\Gamma$ -symmetry of bar-configurations in the global system.

By a simple argument, the above equation can be reduced to the following system of equations:

$$\langle \tilde{q}(e, i) - \psi_e \tilde{q}(e, j), \tilde{q}(e, i) - \psi_e \tilde{q}(e, j) \rangle = \ell_e \quad \forall \Gamma e = (\Gamma i, \Gamma j) \in E(H/\Gamma), \quad (57)$$

where  $\psi(\Gamma e) = \psi_e$  denotes its gain in the quotient  $\Gamma$ -gain graph of  $H$ .

Thus, we may consider the problem in a general  $\Gamma$ -gain graph  $(G = (V, E), \psi)$ . Namely, given a  $\Gamma$ -gain graph  $(G = (V, E), \psi)$ ,  $B(v) = (A_v, p_v)$  for  $v \in V(G)$ , and  $q_{e,i}, q_{e,j} \in \mathbb{R}^d$  for  $e = (i, j) \in E(G)$ , we shall consider

$$\langle \tilde{q}_{e,i} - \psi_e \cdot \tilde{q}_{e,j}, \tilde{q}_{e,i} - \psi_e \cdot \tilde{q}_{e,j} \rangle = \ell_e \quad \forall e = (i, j) \in E(G). \quad (58)$$

Let us focus on the equation for  $e = (i, j) \in E(G)$  and simply denote  $q_{e,i}$  by  $q_i$ . If we rewrite (58) for  $e$ , we have

$$\langle A_i q_i + p_i - (A_{\psi_e} A_j q_j + A_{\psi_e} p_j + t_{\psi_e}), A_i q_i + p_i - (A_{\psi_e} A_j q_j + A_{\psi_e} p_j + t_{\psi_e}) \rangle = \ell_e.$$

We now investigate infinitesimal motions of bodies under bar-constraints. To do that, we take the derivative with respect to  $(A_v, p_v)$  for  $v \in V$ , leading to

$$\langle A_i q_i + p_i - (A_{\psi_e} A_j q_j + A_{\psi_e} p_j + t_{\psi_e}), \dot{A}_i q_i + \dot{p}_i - (A_{\psi_e} \dot{A}_j q_j + A_{\psi_e} \dot{p}_j) \rangle = 0. \quad (59)$$

Without loss of generality, we may take  $(A_v, p_v) = (I_d, 0)$  for all  $v \in V(G)$ . Then,  $\dot{A}_v$  is a  $d \times d$  skew-symmetric matrix. Namely, (59) is a linear equation of variables  $(\dot{A}_i, \dot{p}_i)$  and  $(\dot{A}_j, \dot{p}_j)$  written by

$$\langle q_i - (A_{\psi_e} q_j + t_{\psi_e}), \dot{A}_i q_i + \dot{p}_i - (A_{\psi_e} \dot{A}_j q_j + A_{\psi_e} \dot{p}_j) \rangle = 0. \quad (60)$$

It is known that, for any  $h, q \in \mathbb{R}^d$  and a  $d \times d$  skew-symmetric matrix  $\dot{A}$ , there is a  $\binom{d}{2}$ -dimensional vector  $\omega$  such that  $\langle h, \dot{A}q \rangle = \langle h \wedge q, \omega \rangle$ , obtained by aligning  $\binom{d}{2}$  independent entries of  $\dot{A}$  in an appropriate manner. Therefore, replacing  $\dot{A}_i$  and  $\dot{A}_j$  with the corresponding  $\binom{d}{2}$ -dimensional vectors  $\omega_i$  and  $\omega_j$ , we have the following two relations to simplify (60):

$$\begin{aligned} \langle q_i - A_{\psi_e} q_j - t_{\psi_e}, \dot{A}_i q_i \rangle &= \langle (q_i - A_{\psi_e} q_j - t_{\psi_e}) \wedge q_i, \omega_i \rangle = -\langle (A_{\psi_e} q_j + t_{\psi_e}) \wedge q_i, \omega_i \rangle \\ \langle q_i - A_{\psi_e} q_j - t_{\psi_e}, A_{\psi_e} \dot{A}_j q_j \rangle &= \langle (A_{\psi_e}^{-1}(q_i - A_{\psi_e} q_j - t_{\psi_e})) \wedge q_j, \omega_j \rangle = \langle (A_{\psi_e}^{-1}(q_i - t_{\psi_e})) \wedge q_j, \omega_j \rangle. \end{aligned}$$

Thus, (60) can be written by

$$\langle q_i - (A_{\psi_e} q_j + t_{\psi_e}), \dot{p}_i \rangle - \langle A_{\psi_e}^{-1}(q_i - t_{\psi_e}) - q_j, \dot{p}_j \rangle - \langle (A_{\psi_e} q_j + t_{\psi_e}) \wedge q_i, \omega_i \rangle - \langle (A_{\psi_e}^{-1}(q_i - t_{\psi_e})) \wedge q_j, \omega_j \rangle = 0$$

or equivalently,

$$\langle q_i - \psi_e q_j, \dot{p}_i \rangle - \langle \psi_e^{-1} q_i - q_j, \dot{p}_j \rangle - \langle (\psi_e q_j) \wedge q_i, \omega_i \rangle - \langle (\psi_e^{-1} q_i) \wedge q_j, \omega_j \rangle = 0. \quad (61)$$

The pair  $(\omega_j, \dot{p}_j)$  is conventionally called a *screw motion* and the set of all screw motions forms a  $\binom{d+1}{2}$ -dimensional linear space, which can be identified with  $\bigwedge^2 \mathbb{R}^{d+1}$ .

For  $q \in \mathbb{R}^d$ , denote  $q^\uparrow = \begin{pmatrix} q \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1}$ . Since  $q_1^\uparrow \wedge q_2^\uparrow = (q_1 \wedge q_2, q_1 - q_2)$  for any  $q_1, q_2 \in \mathbb{R}^d$ , if we denote the pair  $(\omega_j, \dot{p}_j)$  as a  $\binom{d+1}{2}$ -dimensional vector  $s_j \in \bigwedge^2 \mathbb{R}^{d+1}$ , (61) is further converted to

$$\langle q_i^\uparrow \wedge \rho(\psi_e) q_j^\uparrow, s_i \rangle - \langle \rho(\psi_e)^{-1} q_i^\uparrow \wedge q_j^\uparrow, s_j \rangle = 0. \quad (62)$$

Thus, our problem is formulated as follows: for a  $\Gamma$ -gain graph  $(G, \psi)$  and  $q_{e,i}, q_{e,j} \in \mathbb{R}^d$  for each  $e = (i, j) \in E(G)$ , an *infinitesimal motion* is defined by  $s : i \in V(G) \mapsto s_i \in \bigwedge^2 \mathbb{R}^{d+1}$  satisfying

$$\langle q_{e,i}^\uparrow \wedge \rho(\psi_e) q_{e,j}^\uparrow, s(i) \rangle - \langle \rho(\psi_e)^{-1} q_{e,i}^\uparrow \wedge q_{e,j}^\uparrow, s(j) \rangle = 0 \quad \forall e = (i, j) \in E(G), \quad (63)$$

and we are asked to compute the dimension of the space of infinitesimal motions. To do that, we may replace  $q_{e,i}^\uparrow$  by  $\rho(\psi_e) q_{e,i}^\uparrow$  and consider the following system of equations:

$$\langle \rho(\psi_e) q_{e,i}^\uparrow \wedge \rho(\psi_e) q_{e,j}^\uparrow, s(i) \rangle - \langle q_{e,i}^\uparrow \wedge q_{e,j}^\uparrow, s(j) \rangle = 0 \quad \forall e = (i, j) \in E(G). \quad (64)$$

Define  $\hat{x}_{e,\psi}^{(2)}$  by

$$\hat{x}_{e,\psi}^{(2)}(v) = \begin{cases} -\rho(\psi_e) q_{e,i}^\uparrow \wedge \rho(\psi_e) q_{e,j}^\uparrow & \text{if } v = i \\ q_{e,i}^\uparrow \wedge q_{e,j}^\uparrow & \text{if } v = j \\ 0 & \text{otherwise} \end{cases}$$

for a non-loop edge  $e = (i, j)$ , and for each loop attached to a vertex  $i$

$$\hat{x}_{e,\psi}^{(2)}(v) = \begin{cases} \rho(\psi_e) q_{e,i}^\uparrow \wedge \rho(\psi_e) q_{e,j}^\uparrow - q_{e,i}^\uparrow \wedge q_{e,j}^\uparrow & \text{if } v = i \\ 0 & \text{otherwise.} \end{cases}$$

Observe that  $s$  is a solution of (64) if and only if  $s$  is in the orthogonal complement of  $\text{span}\{\hat{x}_{e,\psi}^{(2)} \mid e \in E(G)\}$ . However, since  $\hat{x}_{e,\psi}^{(2)}$  is a special case of (51)(52) given in the last subsection, we can apply Theorem 7.1 to compute the exact value of  $\dim(\text{span}\{\hat{x}_{e,\psi}^{(2)} \mid e \in E(G)\})$  if a bar-configuration  $q$  is generic. (Although the last coordinate is restricted to 1 in each  $q_{e,i}^\uparrow$ , we can still apply Theorem 7.1, as  $\dim(\text{span}\{\hat{x}_{e,\psi}^{(2)} \mid e \in E(G)\})$  is invariant up to scalar multiples of  $q_{e,i}^\uparrow$ .)

In terms of the infinitesimal rigidity of  $\Gamma$ -symmetric body-bar frameworks, we proved the following:

**Theorem 7.2.** *Let  $H$  be a  $\Gamma$ -symmetric graph. Then, for almost all body-configurations  $B$  and bar-configurations  $q$ , the  $\Gamma$ -symmetric body-bar framework  $(H, B, q)$  is infinitesimally rigid if and only if the quotient  $\Gamma$ -gain graph contains an edge subset  $I$  satisfying the following:*

- $|I| = D|V| - D + \dim(\text{span}\{\text{im}(I_D - \rho^{(2)}(\gamma)) \mid \gamma \in \Gamma\})$ ;
- for any  $F \subseteq I$ ,  $|F| \leq D|V(F)| - Dc(F) + \sum_{X \in C(F)} \dim(\text{span}\{\text{im}(I_D - \rho^{(2)}(\gamma)) \mid \gamma \in \langle X \rangle\})$

where  $D = \binom{d+1}{2}$ ,  $\rho : \Gamma \rightarrow GL(\mathbb{R}^{d+1})$  is a linear representation of  $\Gamma$  by augmented  $(d+1) \times (d+1)$ -matrices (53), and  $\rho^{(2)} : \Gamma \rightarrow GL(\wedge^2 \mathbb{R}^{d+1})$  is a linear representation of  $\Gamma$  defined by  $\rho^{(2)}(\gamma)(v_1 \wedge v_2) = \rho(\gamma)v_1 \wedge \rho(\gamma)v_2$  for  $v_1, v_2 \in \mathbb{R}^{d+1}$  and  $\gamma \in \Gamma$ .

As a special case when  $\Gamma$  is a lattice, Theorem 7.2 verifies a conjecture by Ross [27].

## 8 Generalization of Lift Matroids

In [43], Zaslavsky also introduced another matroid of gain graphs, called *lift matroids*. This matroid is a special case of *elementary lifts* of graphic matroids, (see e.g., [25] for elementary lifts). It was shown by Zaslavsky [44] that a lift matroid is representable over  $\mathbb{F}$  if the underlying gain group is isomorphic to an additive subgroup of  $\mathbb{F}$ . In this section, we shall propose an extension of lift matroids.

### 8.1 Lift matroids

Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph. In the *lift matroid*  $\mathbf{L}(G, \psi)$  of  $(G, \psi)$ ,  $F \subseteq E$  is independent if and only if there is at most one cycle, which is unbalanced if exists [43]. Therefore, if we define  $\ell_\Gamma : 2^E \rightarrow \mathbb{Z}$  by

$$\ell_\Gamma(F) = |V(F)| - c(F) + \alpha_\Gamma(F) \quad (F \subseteq E), \quad (65)$$

where  $\alpha_\Gamma$  is as defined in (12), then  $\ell_\Gamma$  is the rank function of  $\mathbf{L}(G, \psi)$ .

Suppose that  $\Gamma$  is an additive subgroup of  $\mathbb{F}$ . We shall add a special new sign  $*$  to  $V$ , and consider a linear representation given by  $e \in E(G) \mapsto L_e \subseteq \mathbb{F}^{V \cup \{*\}}$  with

$$L_e = \left\{ x \in \mathbb{F}^{V \cup \{*\}} \left| \begin{array}{l} x(i) + x(j) = 0, \\ \psi(e)x(i) + x(*) = 0, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right. \right\}.$$

This gives a linear representation of  $\mathbf{L}(G, \psi)$ , called the *canonical representation* of  $\mathbf{L}(G, \psi)$  [44, Theorem 4.1]. It is also known that any representation of  $\mathbf{L}(K_n^\bullet, \psi^\bullet)$  is of the form (see [44, §4] for more detail).

### 8.2 Generalized lift matroids

The idea of our extension of lift matroids is the same as the case of gain matroids; instead of  $\alpha_\Gamma(F)$ , we consider a submodular function over  $\Gamma$ .

Suppose that  $(G = (V, E), \psi)$  is a  $\Gamma$ -gain graph with an abelian group  $\Gamma$ . We consider a symmetric polymatroidal function  $\mu : 2^\Gamma \rightarrow \mathbb{R}_+$  over  $\Gamma$  (see §4.2 for definition). Since  $\Gamma$  is abelian, we automatically have the invariance under conjugates.

For  $F \subseteq E$ , we define  $\langle\langle F \rangle\rangle$  by

$$\langle\langle F \rangle\rangle = \langle \psi(W) \mid W \in \pi(F, v), v \in V \rangle,$$



i.e., the group generated by gains of all closed walks in  $F$ . Since  $\Gamma$  is abelian, if  $F$  is connected,  $\langle\langle F \rangle\rangle = \langle F \rangle_v$  for any  $v \in V(F)$  by Proposition 2.1.

We then define  $\ell_\mu : 2^E \rightarrow \mathbb{R}$  by

$$\ell_\mu(F) = |V(F)| - c(F) + \mu\langle\langle F \rangle\rangle \quad (F \subseteq E), \quad (66)$$

where  $\mu\langle\langle F \rangle\rangle$  is an abbreviation of  $\mu(\langle\langle F \rangle\rangle)$ . As in Theorem 4.1, we have the following:

**Theorem 8.1.** *Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph with an abelian group  $\Gamma$ , and  $\mu$  be a symmetric polymatroidal function over  $\Gamma$ . If  $\mu(\gamma) \leq 1$  for every  $\gamma \in \Gamma$ ,  $\ell_\mu$  is monotone submodular.*

*Proof.* For each  $X \subseteq E$  and  $e = (i, j) \in E \setminus X$ , let  $\Delta(X, e) = \ell_\mu(X \cup \{e\}) - \ell_\mu(X)$ , and let  $X_i$  and  $X_j$  be the connected components of  $X$  for which  $i \in V(X_i)$  and  $j \in V(X_j)$ , each of which is an empty set if such a component does not exist. By a simple calculation, we have the following relation:

$$\Delta(X, e) = \begin{cases} \mu\langle\langle X \cup \{e\} \rangle\rangle - \mu\langle\langle X \rangle\rangle & \text{if } X_i = X_j \\ \mu\langle\langle X \cup \{e\} \rangle\rangle + 1 - \mu\langle\langle X \rangle\rangle & \text{otherwise.} \end{cases} \quad (67)$$

However, since  $\Gamma$  is abelian, it can be easily checked that  $\langle\langle X \cup \{e\} \rangle\rangle = \langle\langle X \rangle\rangle$  if  $X_i \neq X_j$ . Therefore, we actually have

$$\Delta(X, e) = \begin{cases} \mu\langle\langle X \cup \{e\} \rangle\rangle - \mu\langle\langle X \rangle\rangle & \text{if } X_i = X_j \\ 1 & \text{otherwise.} \end{cases} \quad (68)$$

By (68), the monotonicity of  $\mu$  over  $\Gamma$  implies that  $\Delta(X, e) \geq 0$ , implying the monotonicity of  $\ell_\mu$ .

To see the submodularity, we claim the following:

**Claim 8.2.** *Let  $X \subseteq E$ ,  $e = (i, j) \in E \setminus X$ , and  $F$  a maximal forest in  $X$ . Suppose that  $\psi(e) = \text{id}$  for  $e \in F$ . Then,*

$$\Delta(X, e) = \begin{cases} \mu(\{\psi(f) \mid f \in X \cup \{e\}\}) - \mu(\{\psi(f) \mid f \in X\}) & \text{if } X_i = X_j \\ 1 & \text{otherwise.} \end{cases}$$

Moreover,  $\Delta(X, e) \leq 1$ .

*Proof.* By Proposition 2.4,

$$\langle\langle X \rangle\rangle = \langle \psi(f) \mid f \in X \rangle \quad \text{and} \quad \langle\langle X \cup \{e\} \rangle\rangle = \langle \psi(f) \mid f \in X \cup \{e\} \rangle. \quad (69)$$

By the invariance of  $\mu$  under taking closures, putting (69) into (68), we obtain the former relation of the statement.

To see the latter claim, observe that, if  $X_i = X_j$ ,  $\Delta(X, e) = \mu(\{\psi(f) \mid f \in X \cup \{e\}\}) - \mu(\{\psi(f) \mid f \in X\}) \leq \mu(\psi(e)) - \mu(\emptyset) \leq 1$ , where the second inequality follows from the submodularity of  $\mu$  over  $\Gamma$  and the third one follows from  $\mu(\psi(e)) \leq 1$  and  $\mu(\emptyset) = 0$ .  $\square$

To see the submodularity of  $\ell_\mu$ , let us check  $\Delta(X, e) \geq \Delta(Y, e)$  for any  $X \subseteq Y \subseteq E$  and  $e \in E \setminus Y$ . Since  $\Delta(Y, e) \leq 1$  by Claim 8.2, it suffices to consider the case when  $\Delta(X, e) < 1$ , i.e.,  $X_i = X_j$ . In this case,  $Y_i = Y_j$  as well. Hence,  $\Delta(X, e) \geq \Delta(Y, e)$  directly follows from Claim 8.2 and the submodularity (14) of  $\mu$  over  $\Gamma$ .  $\square$

As in the case of gain matroids, let us focus on rational functions  $\mu$ , i.e.,  $\mu : 2^\Gamma \rightarrow \{0, \frac{k}{d}, \dots, \frac{d-1}{d}k, k\}$  for some positive integers  $d$  and  $k$ . Then,  $d\ell_\mu$  is a normalized integer-valued monotone submodular function, and hence  $(E, d\ell_\mu)$  is a polymatroid.

*Example 8.1.* Let us consider  $\mathbb{Z}^d$ -gain graph  $(G = (V, E), \psi)$ . If we define  $\mu$  by  $\mu(X) = \dim(\text{span}\{X\})$  for  $X \subseteq \Gamma$ , then  $\mu$  is a symmetric polymatroidal function. Hence,  $\ell_\mu(F) = |V(F)| - c(F) + \mu\langle\langle F \rangle\rangle$  is monotone submodular. Since  $\ell_\mu(e) \leq 1$  for  $e \in E$ ,  $\ell_\mu$  is indeed a rank function of a matroid on  $E$ .  $\square$

*Example 8.2.* Let us consider  $\mathbb{Z}^d$ -gain graph  $(G = (V, E), \psi)$ , again. Define  $\mu$  by  $\mu(X) = \dim(\{\alpha \otimes \gamma \mid \alpha \in \mathbb{R}^d, \gamma \in X\})/d$  for  $X \subseteq \Gamma$ , where  $\alpha \otimes \gamma$  denotes the tensor product of  $\alpha$  and  $\gamma$ . Then,  $\mu$  is a symmetric polymatroidal function with  $\mu(\gamma) \leq 1$  for every  $\gamma \in \Gamma$ . Therefore,  $d\ell_\mu(F) = d|V(F)| - dc(F) + d\mu\langle\langle F \rangle\rangle$  is monotone submodular. Actually, the resulting polymatroid is just the sum of  $d$  copies of the matroid given in Example 8.1.  $\square$

*Remark 8.3.* Note that lifting matroids can be defined on  $\Gamma$ -gain graphs of any group  $\Gamma$ , whereas we assumed in the above extension that  $\Gamma$  is abelian. In fact, Theorem 8.1 holds even for non-abelian group  $\Gamma$ , if  $\mu\langle\langle \cdot \rangle\rangle$  is invariant under switchings, which is the case of lifting matroids.  $\square$

### 8.3 Linear representations of generalized lift matroids

We now give an extension of the canonical representation of  $\mathbf{L}(G, \psi)$ . Let  $(G, \psi)$  be a  $\Gamma$ -gain graph, and suppose that  $\Gamma$  is an additive subgroup of a vector space  $\mathbb{F}^t$  over  $\mathbb{F}$ .

For a bilinear map  $b : \mathbb{F}^d \times \mathbb{F}^t \rightarrow \mathbb{F}^k$ , we define  $\mu_b : 2^\Gamma \rightarrow \mathbb{Z}$  as follows:

$$\mu_b(X) = \dim(\text{span}\{b(\alpha, \gamma) \mid \alpha \in \mathbb{F}^d, \gamma \in X\}) \quad (X \subseteq \Gamma). \quad (70)$$

Then, it is easy to check that  $\mu_b$  is a symmetric polymatroid function over  $\Gamma$ . Also, for any  $\gamma \in \Gamma$ , we have  $\mu_b(\gamma) \leq d$ . Therefore, by Theorem 8.1, the following function  $f_b$  induces a polymatroid of a  $\Gamma$ -gain graph  $(G = (V, E), \phi)$ :

$$f_b(F) = d|V(F)| - dc(F) + \mu_b\langle\langle F \rangle\rangle \quad (F \subseteq E). \quad (71)$$

For example, if setting  $b : \mathbb{F} \times \mathbb{F}^d \rightarrow \mathbb{F}^d$  to be  $b : (\alpha, \gamma) \mapsto \alpha\gamma$ , we have the case of Example 8.1.

We now show a linear representation of the (poly)matroid induced by  $f_b$ . With each edge  $e = (i, j) \in E(G)$ , we associate a linear space

$$L_{e, \psi} = \left\{ x \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k \mid \begin{array}{l} x(i) + x(j) = 0, \\ b(x(i), \psi(e)) + x(*) = 0, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\}$$

if  $e$  is not a loop, and

$$L_{e, \psi} = \left\{ x \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k \mid \exists \alpha \in \mathbb{F}^d : \begin{array}{l} x(*) = -b(\alpha, \psi(e)), \\ x(V) = 0 \end{array} \right\}.$$

if  $e$  is a loop, where  $(\mathbb{F}^d)^V \oplus \mathbb{F}^k$  is an abbreviation of  $(\mathbb{F}^d)^V \oplus (\mathbb{F}^k)^{\{*\}}$  used throughout subsequent discussions.

We consider a linear polymatroid induced on  $\{L_{e, \psi} \mid e \in E\}$ . Clearly, it depends on  $\psi$ , but as in Lemma 5.2 the rank of the polymatroid is invariant up to equivalence of  $\psi$ .

**Lemma 8.3.** *Let  $\psi$  and  $\psi'$  be equivalent gain functions. Then,  $\dim(\text{span}\{L_{e,\psi} \mid e \in E\}) = \dim(\text{span}\{L_{e,\psi'} \mid e \in E\})$ .*

*Proof.* It is sufficient to show that the dimension is invariant from any switch operation.

Suppose that  $\psi'$  is obtained from  $\psi$  by a switch operation at  $v$  with  $\gamma \in \Gamma$ . We may assume that all of edges incident to  $v$  is oriented to  $v$ . Then,  $\psi'(e) = \psi(e) - \gamma$  if  $e$  is incident to  $v$ ; otherwise  $\psi'(e) = \psi(e)$ . Note that, since  $\Gamma$  is abelian,  $\psi'(e) = \psi(e)$  for any loop  $e$ .

Consider a bijective linear transformation  $T : (\mathbb{F}^d)^V \oplus \mathbb{F}^k \rightarrow (\mathbb{F}^d)^V \oplus \mathbb{F}^k$  defined by  $T(x)(w) = x(w)$  for  $w \in V$  and  $T(x)(*) = b(x(v), \gamma) + x(*)$  for the special vertex  $*$ . We then have

$$\begin{aligned} TL_{e,\psi} &= \left\{ T(x) \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k \mid \begin{array}{l} b(x(i), \psi(e)) + x(*) = 0, \\ x(i) + x(j) = 0, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\} \\ &= \left\{ y \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k \mid \begin{array}{l} b(y(i), \psi(e)) - b(y(v), \gamma) + y(*) = 0, \\ y(i) + y(j) = 0, \\ y(V \setminus \{i, j\}) = 0 \end{array} \right\}. \end{aligned}$$

Since  $b(y(v), \gamma) = 0$  if  $v \neq i, j$ , we have  $b(y(i), \psi(e)) - b(y(v), \gamma) = b(y(i), \psi'(e))$  for any  $e = (i, j) \in E$ . Thus,  $TL_{e,\psi} = L_{e,\psi'}$ , implying the lemma.  $\square$

The following theorem is a counterpart of Theorem 5.3, whose proof is almost identical.

**Theorem 8.4.** *Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph with an additive subgroup  $\Gamma$  of  $\mathbb{F}^t$ . Define  $f_b$  and  $L_{e,\psi}$  as above. Then,*

$$f_b(E) = \dim(\text{span}\{L_{e,\psi} \mid e \in E\}).$$

*Proof.* Let  $T$  be a maximal forest in  $E$ . By Proposition 2.3 and Lemma 8.3, we may assume that  $\psi(e) = 0$  for  $e \in T$ . Since  $\Gamma$  is abelian, Proposition 2.1 and Proposition 2.4 imply that  $\langle\langle E \rangle\rangle = \langle\psi(e) \mid e \in E \setminus T\rangle$ , and hence

$$f_b(E) = d|V(E)| - dc(E) + \dim(\text{span}\{b(\alpha, \psi(e)) \mid \alpha \in \mathbb{F}^d, e \in E \setminus T\}).$$

Let  $L_T = \text{span}\{L_{e,\psi} \mid e \in T\}$ . By Lemma 5.1, we deduce that (i)  $\dim(L_T) = d|V(E)| - dc(E)$  and (ii) each quotient space  $L_{e,\psi}/L_T$  for  $e \in E \setminus T$  is written by

$$\{x + L_T \mid \exists \alpha \in \mathbb{F}^d: b(\alpha, \psi(e)) + x(*) = 0, x(V) = 0\},$$

which is isomorphic to  $\{b(\alpha, \psi(e)) \mid \alpha \in \mathbb{F}^d\}$ . Therefore,  $\dim(\text{span}\{L_{e,\psi} \mid e \in E\}) = \dim(L_T) + \dim(\text{span}\{L_{e,\psi}/L_T \mid e \in E \setminus T\}) = d|V(E)| - dc(E) + \dim(\text{span}\{b(\alpha, \psi(e)) \mid \alpha \in \mathbb{F}^d, e \in E \setminus T\}) = f_b(E)$ .  $\square$

Let  $(G, \psi)$  be a  $\Gamma$ -gain graph. With each  $e = (i, j) \in E(G)$ , we associate a vector  $y_{e,\psi}$  from  $L_{e,\psi}$  so that  $\{y_{e,\psi} \mid e \in E(G)\}$  is in generic position, by extending the underlying field to  $\mathbb{K}^d$ . The following is an immediate consequence of Theorem 3.1 and Theorem 8.4.

**Corollary 8.5.** *Let  $(G, \psi)$  be a  $\Gamma$ -gain graph with an additive subgroup  $\Gamma$  of  $\mathbb{K}^t$ . Let  $b : \mathbb{K}^d \times \mathbb{F}^t \rightarrow \mathbb{K}^k$  be a bilinear map. Then,  $\{y_{e,\psi} \mid e \in E(G)\}$  is linearly independent if and only if for any  $F \subseteq E$*

$$|F| \leq d|V(F)| - dc(F) + \dim(\text{span}\{b(\alpha, \gamma) \mid \alpha \in \mathbb{K}^d, \gamma \in \langle\langle F \rangle\rangle\}).$$

## 8.4 Applications

Let  $(G, \psi)$  be a  $\mathbb{Z}^d$ -gain graph, and let us define a bilinear map  $b : \mathbb{Q}^d \times \mathbb{Q}^d \rightarrow \mathbb{Q}^{d^2}$  by  $b(\alpha, \gamma) = \alpha \otimes \gamma$ , a tensor product of  $\alpha$  and  $\gamma$ . For each  $e = (i, j) \in E(G)$ , we shall associate a vector  $y_{e, \psi} \in (\mathbb{F}^d)^V \oplus \mathbb{F}^{d^2}$  with

$$y_{e, \psi}(v) = \begin{cases} \alpha_e & \text{if } v = i \\ -\alpha_e & \text{if } v = j \\ -\alpha_e \otimes \psi(e) & \text{if } v = * \\ 0 & \text{otherwise} \end{cases}$$

such that the set of all coordinates of  $\alpha_e$  ( $e \in E(G)$ ) is algebraically independent over  $\mathbb{Q}$ . By Corollary 8.5,  $\{y_{e, \psi} \mid e \in E(G)\}$  is linearly independent if and only if for any  $F \subseteq E$

$$|F| \leq d|V(F)| - dc(F) + d \dim(\text{span}\{\gamma \mid \gamma \in \langle\langle F \rangle\rangle\}).$$

As in § 6, it is easy to check that the restriction of  $\{L_{e, \psi} \mid e \in E\}$  to a generic hyperplane gives rise to the orbit rigidity matrix of a  $\mathbb{Z}^2$ -symmetric framework (called a periodic framework) when  $d = 2$  or to the linear representation of the  $\mathbb{Z}^d$ -symmetric parallel redrawing polymatroid of a  $\mathbb{Z}^d$ -symmetric framework for general dimension  $d$ . This implies that the independence in the associated linear (poly)matroid is characterized by the following counting condition: for any nonempty  $F \subseteq E$

$$|F| \leq d|V(F)| - dc(F) + d \dim(\text{span}\{\gamma \mid \gamma \in \langle\langle F \rangle\rangle\}) - 1.$$

This is an alternative proof of results by Malestein and Theran [20] for  $d = 2$ .

## 9 Toward unified matroids

Although we have no clear idea on how to unify the extension of gain matroids and that of lift matroids via their rank functions, the canonical representations tell us a natural approach to unify representation theory obtained so far. To see this, in this section, we shall focus on subgroups of  $GL(\mathbb{F}^d) \ltimes \mathbb{F}^d$ , that is, the semidirect product of  $GL(\mathbb{F}^d)$  and  $\mathbb{F}^d$  with  $(g, z) \cdot (g', z') = (gg', gz' + z)$ .

Let  $\Gamma$  be a subgroup of  $GL(\mathbb{F}^d) \ltimes \mathbb{F}^d$ . The projection of  $\Gamma$  to the first component, i.e.,  $(g, z) \mapsto g$ , is a group homomorphism, and hence the image  $\{g \mid (g, z) \in \Gamma\}$  forms a subgroup of  $GL(\mathbb{F}^d)$ , called the *linear part* of  $\Gamma$  and denoted by  $\Gamma_1$ .

Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph with a gain function  $\psi = (\psi_1, \psi_2)$ , and let  $b : \mathbb{F}^d \times \mathbb{F}^d \rightarrow \mathbb{F}^k$  be a bilinear map such that  $\Gamma_1$  is unitary with respect to  $b$ , i.e.,  $b(gx, y) = b(x, g^{-1}y)$  for any  $g \in \Gamma_1$ . Combining the idea of §5 and §8, we now associate a linear subspace with each edge  $e = (i, j) \in E$  as follows:

$$U_{e, \psi} = \left\{ x \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k \left| \begin{array}{l} x(i) + \psi_1(e)x(j) = 0, \\ x(*) = -b(x(i), \psi_2(e)), \\ x(V \setminus \{i, j\}) = 0 \end{array} \right. \right\} \quad (72)$$

if  $e$  is not a loop, and

$$U_{e, \psi} = \left\{ x \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k \left| \exists \alpha \in \mathbb{F}^d : \begin{array}{l} x(i) = (I_d - \psi_1(e))\alpha, \\ x(*) = -b(\alpha, \psi_2(e)), \\ x(V \setminus \{i\}) = 0 \end{array} \right. \right\} \quad (73)$$

if  $e$  is a loop attached to  $i$ .

Note that  $U_{e,\psi}$  is invariant from the orientation of  $e$ , as

$$\begin{aligned} U_{e,\psi} &= \{x \mid x(i) + \psi_1(e)x(j) = 0, b(x(i), \psi_2(e)) + x(*) = 0, x(V \setminus \{i, j\}) = 0\} \\ &= \{x \mid \psi_1(e)^{-1}x(i) + x(j) = 0, b(-\psi_1(e)x(j), \psi_2(e)) + x(*) = 0, x(V \setminus \{i, j\}) = 0\} \\ &= \{x \mid \psi_1(e)^{-1}x(i) + x(j) = 0, b(x(j), -\psi_1(e)^{-1}\psi_2(e)) + x(*) = 0, x(V \setminus \{i, j\}) = 0\} \end{aligned}$$

where  $\psi(e)^{-1} = (\psi_1(e)^{-1}, -\psi_1(e)^{-1}\psi_2(e))$ . Although  $U_{e,\psi}$  depends on the choice of gain functions  $\psi$ , as in the previous cases, the rank of the polymatroid induced on the representation is invariant up to equivalence.

**Lemma 9.1.** *Let  $\psi$  and  $\psi'$  be equivalent gain functions. Then,  $\dim\{\text{span}\{U_{e,\psi} \mid e \in E\}\} = \dim\{\text{span}\{U_{e,\psi'} \mid e \in E\}\}$ .*

*Proof.* Suppose that  $\psi'$  is obtained from  $\psi$  by a switch operation at  $v$  with  $\gamma = (g, z) \in \Gamma$ . Since  $U_{e,\psi}$  is invariant from the direction of  $e$ , we may assume that all of the edges incident to  $v$  are oriented from  $v$ . Then,  $\psi'(e) = \gamma\psi(e)$  if  $e$  is a non-loop edge incident to  $v$ ;  $\psi'(e) = \gamma\psi(e)\gamma^{-1}$  if  $e$  is a loop incident to  $v$ ; otherwise  $\psi'(e) = \psi(e)$ .

Consider a bijective linear transformation  $T : (\mathbb{F}^d)^V \oplus \mathbb{F}^k \rightarrow (\mathbb{F}^d)^V \oplus \mathbb{F}^k$  defined by, for each  $x \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k$ ,

$$T(x)(w) = \begin{cases} x(w) & \text{if } w \in V \setminus \{v\} \\ gx(v) & \text{if } w = v \\ -b(x(v), g^{-1}z) + x(*) & \text{if } w = *. \end{cases}$$

We then have

$$\begin{aligned} x(w) &= T(x)(w) \text{ for } w \in V \setminus \{v\}, \\ x(v) &= g^{-1}T(x)(v), \\ x(*) &= T(x)(*) + b(x(v), g^{-1}z) = T(x)(*) + b(T(x)(v), z). \end{aligned}$$

Therefore, if  $e$  is a non-loop edge oriented from  $v$  to  $j \in V$ ,

$$TU_{e,\psi} = \left\{ y \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k \mid \begin{array}{l} y(v) + g\psi_1(e)y(j) = 0, \\ b(y(v), g\psi_2(e) + z) + y(*) = 0, \\ y(V \setminus \{v, j\}) = 0 \end{array} \right\}.$$

As  $\psi'(e) = (g\psi_1(e), g\psi_2(e) + z)$ , we obtain that  $TU_{e,\psi} = U_{e,\psi'}$ . Similarly, for a loop  $e$  attached to  $v$ ,

$$\begin{aligned} TU_{e,\psi} &= \left\{ y \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k \mid \exists \alpha \in \mathbb{R}^d : \begin{array}{l} y(v) = g(I_d - \psi_1(e))\alpha, \\ y(*) = -b(\alpha, -\psi_1(e)^{-1}g^{-1}z + g^{-1}z + \psi_2(e)), \\ y(V \setminus \{v, j\}) = 0 \end{array} \right\} \\ &= \left\{ y \in (\mathbb{F}^d)^V \oplus \mathbb{F}^k \mid \exists \alpha \in \mathbb{R}^d : \begin{array}{l} y(v) = (I_d - g\psi_1(e)g^{-1})\alpha, \\ y(*) = -b(\alpha, -g\psi_1(e)^{-1}g^{-1}z + g\psi_2(e) + z), \\ y(V \setminus \{v, j\}) = 0 \end{array} \right\} \\ &= U_{e,\psi'}. \end{aligned}$$

If  $e = (i, j)$  is not incident to  $v$ , then we have  $T(x)(i) = x(i)$ ,  $T(x)(j) = x(j)$ , and  $T(x)(*) = x(*)$  by  $x(v) = 0$  for any  $x \in U_{e,\psi}$ , and hence  $TU_{e,\psi} = U_{e,\psi} = U_{e,\psi'}$ . Thus, we obtain the lemma.  $\square$

By using Lemma 9.1, we can now apply the same proof as Theorem 5.3 to show a combinatorial characterization of the polymatroid induced on  $\{U_{e,\psi} \mid e \in E\}$ . To see it, we need a new terminology. Consider  $F \subseteq E$ . Recall that  $G[F]$  denotes the edge-induced subgraph  $(V(F), F)$ . By Proposition 2.3, for a maximal forest  $T$  of  $F$ , there is an equivalent labeling  $\psi_F^\circ$  to  $\psi$  such that  $\psi_F^\circ(e) = \text{id}$  for all  $e \in T$ . A *compressed graph* by  $F$  is defined as a  $\Gamma$ -gain graph  $(G_F^\circ, \psi_F^\circ)$ , where  $G_F^\circ$  is obtained from  $G[F]$  by contracting each connected component to a single vertex, where each edge  $e$  of  $F$  remains in  $G_F^\circ$  as a loop with the gain  $\psi_F^\circ(e)$ . By Proposition 2.3 and Proposition 2.4,  $(G_F^\circ, \psi_F^\circ)$  is invariant from the choice of  $T$  up to the equivalence of gain functions  $\psi_F^\circ$ .

Applying the same proof as Theorem 8.4, we now have the following result. We omit the proof, which is identical to those of Theorem 5.3 and Theorem 8.4.

**Theorem 9.2.** *Let  $\Gamma$  be a subgroup of  $GL(\mathbb{F}^d) \ltimes \mathbb{F}^d$ , and  $\Gamma_1$  be the projection of  $\Gamma$  to  $GL(\mathbb{F}^d)$ . Let  $(G, \psi)$  be a  $\Gamma$ -gain graph, and  $b : \mathbb{F}^d \times \mathbb{F}^d \rightarrow \mathbb{F}^k$  a bilinear map such that  $\Gamma_1$  is unitary with respect to  $b$ . Then, for any  $F \subseteq E(G)$ ,*

$$\dim(\text{span}\{U_{e,\psi} \mid e \in F\}) = d|V(F)| - dc(F) + \dim(\text{span}\{U_{e,\psi_F^\circ} \mid e \in E(G_F^\circ)\}).$$

*Remark 9.1.* Theorem 9.2 gives a good characterization of the dimension of  $\text{span}\{U_{e,\psi} \mid e \in F\}$ , since computing  $\dim(\text{span}\{U_{e,\psi_F^\circ} \mid e \in E(G_F^\circ)\})$  can be reduced to the computation of the rank of a matrix of size  $(dc(F) + k) \times d|F|$ . Hence, it is possible to compute  $\dim(\text{span}\{U_{e,\psi} \mid e \in F\})$  deterministically in polynomial time.  $\square$

## 10 Further applications

As applications of Theorem 9.2, we shall extend the result of §6 to symmetric frameworks with crystallographic symmetry.

### 10.1 Space groups

For detailed analysis on frameworks with crystallographic symmetry, let us review fundamentals on space groups.

Recall that a *space group* (or *crystallographic group*)  $\Gamma$  is a discrete cocompact subgroup of the Euclidean group  $\mathcal{E}(d)$ , and each element  $(A, t) \in \Gamma$  acts on  $\mathbb{R}^d$  by  $(A, t) \cdot q = Aq + t$  for  $q \in \mathbb{R}^d$ . An element of the form  $(I_d, t)$  is called a *translation*, and is simply denoted by  $t$ . As in the previous section, let  $\Gamma_1 = \{A_\gamma \mid \gamma \in \Gamma\}$ , the projection to the first component.

The subgroup  $\mathcal{L}$  consisting of all translations in  $\Gamma$  is called the *lattice group* of  $\Gamma$ , and it is known by Bieberbach's theorem that  $\mathcal{L}$  is a normal subgroup of  $\Gamma$  generated by  $d$  linearly independent translations  $t_1, \dots, t_d \in \mathbb{R}^d$ . The  $d \times d$ -matrix  $B_\Gamma$  of the base transformation from the standard basis of  $\mathbb{R}^d$  to  $\{t_1, \dots, t_d\}$  is called a *lattice basis* of  $\Gamma$ . (Conventionally, a lattice basis of  $\mathcal{L}$  means  $\{t_1, \dots, t_d\}$ , rather than  $B_\Gamma$ .) Then,  $\mathcal{L} = \{B_\Gamma z \mid z \in \mathbb{Z}^d\}$ .

The quotient subgroup  $\mathcal{K} = \Gamma/\mathcal{L}$  is known as a *point group* of  $\Gamma$ . Since  $\mathcal{K}$  acts on  $\mathcal{L}$  and  $\mathcal{L}$  is isomorphic to  $\mathbb{Z}^d$ ,  $\mathcal{K}$  can be represented as integral matrices. Therefore, in the subsequent discussion,  $\mathcal{K}$  is regarded as a finite subgroup of  $GL(\mathbb{Z}^d)$ . Note then that  $B_\Gamma \mathcal{K} B_\Gamma^{-1} = \Gamma_1 \subseteq \mathcal{O}(\mathbb{R}^d)$ . Indeed, using the lattice basis  $B_\Gamma$ , each element  $\gamma = (A_\gamma, t_\gamma)$  of  $\Gamma$  can be uniquely written by a triple  $(K_\gamma, z_\gamma, c_\gamma) \in GL(\mathbb{Z}^d) \times \mathbb{Z}^d \times [0, 1)^d$ , where  $A_\gamma = B_\Gamma K_\gamma B_\Gamma^{-1}$  and  $t_\gamma = B_\Gamma(z_\gamma + c_\gamma)$ . The representation in  $GL(\mathbb{Z}^d) \times \mathbb{Z}^d \times [0, 1)^d$  is sometimes

called the *standard form*. Note that a space group  $\Gamma$  is determined by the standard form of each element and the lattice basis.

A space group  $\Gamma$  is called *symmorphic* if  $c_\gamma = 0$  for all  $\gamma \in \Gamma$ . If  $\Gamma$  is symmorphic, we may write the standard form of each element by a pair in  $GL(\mathbb{Z}^d) \times \mathbb{Z}^d$ .

Two space groups  $\Gamma$  and  $\Gamma'$  are called *equivalent* if they are conjugate via an affine transformation in  $\text{Aff}(\mathbb{R}^d)$ . Keeping the origin fixed, such an affine motion changes the lattice basis  $B_\Gamma$  without changing the linear part. A motion of the space group among an equivalence class can be thus regarded as a motion of the lattice basis. We hence define the *space of lattices* by

$$\text{Lat}(\Gamma) = \{B \in GL(\mathbb{R}^d) \mid \forall K \in \mathcal{K}: B_\Gamma K B_\Gamma^{-1} = B K B^{-1}\}. \quad (74)$$

It is convenient to consider a slightly larger set

$$\overline{\text{Lat}}(\Gamma) = \{B \in \mathbb{R}^{d \times d} \mid \forall K \in \mathcal{K}: B_\Gamma K B_\Gamma^{-1} = B K B^{-1}\}. \quad (75)$$

Then,  $\overline{\text{Lat}}(\Gamma)$  is a linear space and  $\text{Lat}(\Gamma)$  is a dense open subset of  $\overline{\text{Lat}}(\Gamma)$ . For  $B \in \overline{\text{Lat}}(\Gamma) \setminus \text{Lat}(\Gamma)$ ,  $\{(BK_\gamma B^{-1}, B(z_\gamma + c_\gamma)) \mid \gamma \in \Gamma\}$  forms a lower dimensional space group, which we called a *collapse* of  $\Gamma$ .

$\dim(\overline{\text{Lat}}(\Gamma)) + 2$  is said to be the *degree of freedom of the lattice* of  $\Gamma$ , where the additional term expresses the degree of freedom of translations of the lattice (i.e., the degree of freedom of the origin).

## 10.2 Parallel redrawing with symmorphic space group symmetry

Let us move to the  $\Gamma$ -symmetric parallel redrawing problem for a symmorphic space group  $\Gamma$ . Let  $\mathcal{L}$  be the lattice group of  $\Gamma$  with a basis  $B_\Gamma \in GL(\mathbb{R}^d)$ . Each element  $\gamma \in \Gamma$  is denoted by  $(A_\gamma, t_\gamma) \in \mathcal{O}(\mathbb{R}^d) \ltimes \mathbb{R}^d$ , but we also use the standard form  $(K_\gamma, z_\gamma) \in GL(\mathbb{Z}^d) \times \mathbb{Z}^d$ , where  $A_\gamma = B_\Gamma K_\gamma B_\Gamma^{-1}$  and  $t_\gamma = B_\Gamma z_\gamma$ . Also, let  $\Gamma_1 = \{A_\gamma \mid \gamma \in \Gamma\}$ .

We consider a  $\Gamma$ -symmetric framework  $(H, p)$ , where  $H$  is a  $\Gamma$ -symmetric graph and  $p$  is a  $\Gamma$ -symmetric point-configuration.  $(H, q)$  is said to be a *symmetric parallel redrawing* of  $(H, p)$  if  $(H, q)$  is a  $\Gamma'$ -symmetric parallel redrawing of  $(H, p)$  for some equivalent group  $\Gamma'$  to  $\Gamma$  or a collapse  $\Gamma'$  of  $\Gamma$ .

A *relocation* of  $(H, p)$  is  $m : V(H) \rightarrow \mathbb{R}^d$  such that

$$m(i) - m(j) \text{ is parallel to } p(i) - p(j) \text{ for any } \{i, j\} \in E(H). \quad (76)$$

Regarding the  $\Gamma$ -symmetry of  $m$ , we have two remarks: Since  $m$  is a vector, rather than a point in the Euclidean space, only  $\Gamma_1$  acts on the space of relocations; A framework can be also relocated by deforming the underlying lattice. Thus, we say that a relocation  $m$  is  *$\Gamma$ -symmetric* if there is  $M \in \overline{\text{Lat}}(\Gamma) - B_\Gamma$  such that

$$m(\gamma v) = A_\gamma m(v) + M z_\gamma \quad \forall v \in V(H), \forall \gamma \in \Gamma. \quad (77)$$

The definition is justified by the following proposition.

**Proposition 10.1.** *Let  $(H, p)$  be a  $\Gamma$ -symmetric framework with a symmorphic space group  $\Gamma$ . If  $m$  is a  $\Gamma$ -symmetric relocation of  $(H, p)$ , then  $(H, p + m)$  is a symmetric parallel redrawing of  $(H, p)$ . Conversely, if  $(H, q)$  is a symmetric parallel redrawing, then  $q - p$  is a  $\Gamma$ -symmetric relocation.*

*Proof.* Suppose that  $m$  is  $\Gamma$ -symmetric relocation. Define  $q$  by  $q(v) = p(v) + m(v)$  for  $v \in V(H)$ . Since  $m$  is a relocation,  $(H, q)$  is a parallel drawing of  $(H, p)$  by (76). Also, since  $m$  is  $\Gamma$ -symmetric, there exists  $M \in \overline{\text{Lat}}(\Gamma) - B_\Gamma$  for which (77) is satisfied. Let  $B = M + B_\Gamma$ . Then, for any  $v \in V(H)$  and  $\gamma \in \Gamma$ ,

$$\begin{aligned} q(\gamma v) &= p(\gamma v) + m(\gamma v) = A_\gamma p(v) + B_\Gamma z_\gamma + A_\gamma m(v) + M z_\gamma \\ &= A_\gamma(p(v) + m(v)) + B z_\gamma = A_\gamma q(v) + B z_\gamma. \end{aligned}$$

Since  $B \in \overline{\text{Lat}}(\Gamma)$ , this implies that  $(H, q)$  is  $\Gamma'$ -symmetric for an equivalent  $\Gamma'$  to  $\Gamma$  or a collapse  $\Gamma'$  of  $\Gamma$ .

Conversely, suppose that  $(H, q)$  is a symmetric parallel redrawing of  $(H, p)$ . Then,  $(H, q)$  is  $\Gamma'$ -symmetric for some equivalent  $\Gamma'$  to  $\Gamma$  or a collapse  $\Gamma'$  of  $\Gamma$ . This means that there is  $B \in \overline{\text{Lat}}(\Gamma)$  such that  $B$  is a lattice basis of  $\Gamma'$ . Setting  $m = q - p$  and  $M = B - B_\Gamma$ , we see that  $M \in \overline{\text{Lat}}(\Gamma) - B_\Gamma$  and for any  $v \in V(H)$  and  $\gamma \in \Gamma$ ,

$$\begin{aligned} m(\gamma v) &= q(\gamma v) - p(\gamma v) = B K_\gamma B^{-1} q(v) + B z_\gamma - (B_\Gamma K_\gamma B_\Gamma^{-1} p(v) + B_\Gamma z_\gamma) \\ &= B_\Gamma K_\gamma B_\Gamma^{-1} (q(v) - p(v)) + (B - B_\Gamma) z_\gamma \\ &= A_\gamma m(v) + M z_\gamma, \end{aligned}$$

implying that  $m$  is a  $\Gamma$ -symmetric relocation of  $(H, p)$ .  $\square$

As in the case of point group symmetry, a relocation  $m$  is said to be *trivial* if  $m$  is a linear combination of translations  $m_t$  and a dilation  $m_{\text{di}}$ . Indeed, for any  $t \in \bigcap_{\gamma \in \Gamma} (A_\gamma - I_d)$ , translation  $m_t$  defined by  $m_t(v) = t$  for  $v \in V(H)$  is a  $\Gamma$ -symmetric relocation with  $M = 0$ ; on the other hand, a dilation  $m_{\text{di}}$  defined by  $m_{\text{di}}(v) = p(v)$  for  $v \in V(H)$  is also a  $\Gamma$ -symmetric relocation with  $M = B_\Gamma$ .

Motivated by Proposition 10.1, a  $\Gamma$ -symmetric  $(H, p)$  is said to be *symmetrically robust* if all possible  $\Gamma$ -symmetric relocations are trivial.

Let us then show that checking the robustness can be reduced to computing the rank of linear polymatroids of quotient gain graphs, as in the case of point-group symmetry. As before, we first simplify the system (76). Recall that each edge orbit is written by  $\Gamma e = \{\{\gamma i, \gamma \psi_e j\} \mid \gamma \in \Gamma\}$  with  $\psi_e = \psi_e = (A_{\psi_e}, z_{\psi_e})$ . Thus, for each edge  $(\gamma i, \gamma \psi_e j)$  in an edge orbit  $\Gamma e$ , (76) is written by

$$\langle m(\gamma i) - m(\gamma \psi_e j), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p(\gamma i) - p(\gamma \psi_e j), \alpha \rangle = 0.$$

These are indeed equivalent to one equation,

$$\langle m(i) - m(\psi_e j), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p(i) - p(\psi_e j), \alpha \rangle = 0,$$

which is further converted to, by (77),

$$\langle m(i) - (A_{\psi_e} m(j) + M z_{\psi_e}), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p(i) - (A_{\psi_e} p(j) + t_{\psi_e}), \alpha \rangle = 0.$$

for some  $M \in \overline{\text{Lat}}(\Gamma) - B_\Gamma$ .

Therefore, the problem can be considered in a general  $\Gamma$ -gain graph  $(G = (V, E), \psi)$ , and it suffices to analyze the space of  $(m, M) \in (\mathbb{R}^d)^V \oplus (\overline{\text{Lat}}(\Gamma) - B_\Gamma)$  satisfying

$$\langle m(i) - (A_{\psi_e} m(j) + M z_{\psi_e}), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p(i) - (A_{\psi_e} p(j) + t_{\psi_e}), \alpha \rangle = 0. \quad (78)$$



for every  $e = (i, j) \in E$ . For further analysis, we shall take a basis  $B_1, \dots, B_k \in \mathbb{R}^{d \times d}$  of  $\overline{\text{Lat}}(\Gamma)$ . We then define a bilinear function  $b_i : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$b_i(\alpha, t) = \langle \alpha, B_i B_\Gamma^{-1} t \rangle \quad ((\alpha, t) \in \mathbb{R}^d \times \mathbb{R}^d). \quad (79)$$

Then, a bilinear map  $b : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^k$  is defined by  $b(\alpha, t) = (b_1(\alpha, t), \dots, b_k(\alpha, t))^\top$ .

Observe that  $\Gamma_1$  is unitary with respect to  $b_i$ . Indeed, for each  $\gamma \in \Gamma$ , we have  $A_\gamma = B_\Gamma K_\gamma B_\Gamma^{-1} = B_i K_\gamma B_i^{-1}$  by definition, and hence  $b_i(A_\gamma \alpha, t) = \langle A_\gamma \alpha, B_i B_\Gamma^{-1} t \rangle = \langle \alpha, A_\gamma^{-1} B_i B_\Gamma^{-1} t \rangle = \langle \alpha, B_i K_\gamma^{-1} B_\Gamma^{-1} t \rangle = \langle \alpha, B_i B^{-1} A_\gamma^{-1} t \rangle = b_i(\alpha, A_\gamma^{-1} t)$ . Hence,  $\Gamma_1$  is also unitary with respect to  $b$ .

We shall associate a  $(d-1)$ -dimensional linear subspace  $P'_{e,\psi}(p)$  with each edge  $e = (i, j) \in E$ , defined by

$$P'_{e,\psi}(p) = U_{e,\psi} \cap \{x \in (\mathbb{R}^d)^V \oplus \mathbb{R}^k \mid \langle p(i) - (A_{\psi_e} p(j) + t_{\psi_e}), x(i) \rangle = 0\} \quad (80)$$

where  $U_{e,\psi}$  is, as defined in (72)(73),

$$U_{e,\psi} = \left\{ x \in (\mathbb{R}^d)^V \oplus \mathbb{R}^k \mid \begin{array}{l} x(i) + A_{\psi_e} x(j) = 0, \\ b(x(i), t_{\psi_e}) + x(*) = 0, \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\} \quad (81)$$

or

$$U_{e,\psi} = \left\{ x \in (\mathbb{R}^d)^V \oplus \mathbb{R}^k \mid \exists \alpha \in \mathbb{R}^d : \begin{array}{l} x(i) = (I_d - A_{\psi_e})\alpha, \\ x(*) = -b(\alpha, t_{\psi_e}), \\ x(V \setminus \{i, j\}) = 0 \end{array} \right\} \quad (82)$$

depending on whether  $e$  is a non-loop or a loop, respectively.

**Lemma 10.2.** *Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph with a symmorphic space group  $\Gamma$ . Then, the dimension of the space of  $(m, M) \in (\mathbb{R}^d)^V \oplus (\overline{\text{Lat}}(\Gamma) - B_\Gamma)$  satisfying (78) is equal to*

$$d|V| + k - \dim(\text{span}\{P'_{e,\psi}(p) \mid e \in E\})$$

where  $k = \dim(\overline{\text{Lat}}(\Gamma))$ .

*Proof.* Note that the space of  $(m, M) \in (\mathbb{R}^d)^V \oplus (\overline{\text{Lat}}(\Gamma) - B_\Gamma)$  satisfying (78) is isomorphic to the space of  $(m, B) \in (\mathbb{R}^d)^V \oplus \overline{\text{Lat}}(\Gamma)$  satisfying

$$\langle m(i) - (A_{\psi_e} m(j) + B z_{\psi_e}), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p(i) - (A_{\psi_e} p(j) + t_{\psi_e}), \alpha \rangle = 0 \quad (83)$$

for every  $e = (i, j) \in E$ . Since  $\{B_1, \dots, B_k\}$  is a basis of  $\overline{\text{Lat}}(\Gamma)$ ,  $\overline{\text{Lat}}(\Gamma)$  is parameterized by  $k$  parameters  $a = (a_1, \dots, a_k)^\top \in \mathbb{R}^k$ . In other words, the space of  $(m, L)$  satisfying (83) is isomorphic to the space of  $(m, a) \in (\mathbb{R}^d)^V \oplus \mathbb{R}^k$  satisfying

$$\langle m(i) - (A_{\psi_e} m(j) + \sum_{\ell} a_\ell B_\ell z_{\psi_e}), \alpha \rangle = 0 \quad \forall \alpha \in \mathbb{R}^d \text{ such that } \langle p(i) - (A_{\psi_e} p(j) + t_{\psi_e}), \alpha \rangle = 0 \quad (84)$$

for every  $e = (i, j) \in E$ .

Observe then that  $(m, a) \in (\mathbb{R}^d)^V \oplus \mathbb{R}^k$  satisfies (84) if and only if  $(m, a)$  is in the orthogonal complement of  $\text{span}\{P'_{e,\psi}(p) \mid e \in E\}$ , because, for any  $x \in P'_{e,\psi}(p)$ , we have

$$\begin{aligned}
\langle (m, a), x \rangle &= \langle m(i), x(i) \rangle + \langle m(j), x(j) \rangle + \langle a, x(*) \rangle \\
&= \langle m(i), x(i) \rangle - \langle m(j), A_{\psi_e}^{-1}x(i) \rangle - \langle a, b(x(i), t_{\psi_e}) \rangle \\
&= \langle m(i) - A_{\psi_e}m(j), x(i) \rangle - \sum_{\ell} a_{\ell} b_{\ell}(x(i), t_{\psi_e}), \\
&= \langle m(i) - A_{\psi_e}m(j), x(i) \rangle - \sum_{\ell} a_{\ell} \langle B_{\ell} z_{\psi_e}, x(i) \rangle \\
&= \langle m(i) - A_{\psi_e}m(j) - \sum_{\ell} a_{\ell} B_{\ell} z_{\psi_e}, x(i) \rangle
\end{aligned}$$

with  $\langle p(i) - (A_{\psi_e}p(j) + t_{\psi_e}), x(i) \rangle = 0$ , where we used  $t_{\psi_e} = B_{\Gamma} z_{\psi_e}$ . This completes the proof.  $\square$

Since the set of trivial relocations forms a linear space of dimension  $\dim(\bigcap_{\gamma \in \Gamma} \ker(A_{\gamma} - I_d)) + 1$ , Lemma 10.2 implies the following.

**Corollary 10.3.** *Let  $(H, p)$  be a  $\Gamma$ -symmetric framework with a symmorphic space group  $\Gamma$ , and  $(H/\Gamma, \psi)$  be the quotient  $\Gamma$ -gain graph of  $H$ . Then,  $(H, p)$  is symmetrically robust if and only if*

$$\dim(\text{span}\{P'_{e,\psi}(p/\Gamma) \mid e \in E(H/\Gamma)\}) = d|V/\Gamma| + k - 1 - \dim\left(\bigcap_{\gamma \in \Gamma} \ker(A_{\gamma} - I_d)\right).$$

### 10.2.1 Combinatorial characterization

By Corollary 10.3, it now suffices to analyze the  $\Gamma$ -symmetric parallel redrawing polymatroid of a  $\Gamma$ -gain graph  $(G = (V, E), \psi)$ , that is, a linear polymatroid with linear representation  $e \mapsto P'_{e,\psi}(p)$ . The following theorem provides a combinatorial characterization of this polymatroid.

We say that the lattice of  $\Gamma$  is *generic* if  $B_{\Gamma}$  is expressed by  $B_{\Gamma} = \sum_{i=1}^k s_i B_i$  such that  $\{s_1, \dots, s_k\}$  is algebraically independent over  $\mathbb{Q}_{\Gamma_1}$ . For a discrete point group  $\mathcal{P}$ , almost all space groups  $\Gamma$  with  $\Gamma_1 = \mathcal{P}$  have generic lattices.

**Theorem 10.4.** *Let  $(G = (V, E), \psi)$  be a  $\Gamma$ -gain graph for a symmorphic space group  $\Gamma$  with a generic lattice, and  $k = \dim(\overline{\text{Lat}}(\Gamma))$ . Define  $h_{\Gamma}$  by*

$$h_{\Gamma}(F) = d|V(F)| - dc(F) + \dim(\text{span}\{U_{e,\psi_F^{\circ}} \mid e \in E(G_F^{\circ})\}) - 1,$$

where  $(G_F^{\circ}, \psi_F^{\circ})$  is the compressed graph of  $(G, \psi)$  by  $F$  (defined in §9). Then, for almost all  $p : V \rightarrow \mathbb{R}^d$ ,

$$\dim(\text{span}\{P'_{e,\psi}(p) \mid e \in E\}) = \hat{h}_{\Gamma}(E).$$

In other words, the  $\Gamma$ -symmetric parallel redrawing polymatroid is equal to the polymatroid induced by  $h_{\Gamma}$ .

*Proof.* Let  $h'_\Gamma = h_\Gamma + 1$ . Note that the linear polymatroid  $\mathbf{LP}(E, \Psi)$  of the linear representation  $\Psi : e \mapsto U_{e,\psi}$  is a special case of those given in §9, and Theorem 9.2 implies that  $\mathbf{LP}(E, \Psi) = \mathbf{P}(h'_\Gamma)$ , which is the polymatroid induced by  $h'_\Gamma$ .

Since the lattice of  $\Gamma$  is generic, a lattice basis  $B_\Gamma$  is written by  $B_\Gamma = \sum_{i=1}^k s_i B_i$ , where  $\{s_1, \dots, s_k\}$  is algebraically independent over  $\mathbb{Q}_{\Gamma_1}$ . Let us take any  $p : V \rightarrow \mathbb{R}^d$  such that the coordinates of the image and  $s_1, \dots, s_k$  are algebraically independent over  $\mathbb{Q}_{\Gamma_1}$ . We define a hyperplane  $H$  of  $(\mathbb{R}^d)^V \oplus \mathbb{R}^k$  by

$$H = \{x \in (\mathbb{R}^d)^V \oplus \mathbb{R}^k \mid \sum_{v \in V} \langle p(v), x(v) \rangle + \langle s, x(*) \rangle = 0\}.$$

Then, observe that  $P'_{e,\psi}(p) = U_{e,\psi} \cap H$  for every  $e \in E$ , since, for any  $e = (i, j) \in E$  and any  $x \in U_{e,\psi}$ , we have

$$\begin{aligned} \sum_{v \in V} \langle p(v), x(v) \rangle + \langle s, x(*) \rangle &= \langle p(i), x(i) \rangle + \langle p(j), x(j) \rangle + \langle s, x(*) \rangle \\ &= \langle p(i), x(i) \rangle + \langle p(j), -A_{\psi_e}^{-1} x(i) \rangle + \langle s, -b(x(i), t_{\psi_e}) \rangle \\ &= \langle p(i), x(i) \rangle - \langle A_{\psi_e} p(j), x(i) \rangle - \sum_{\ell} s_\ell \langle B_\ell z_{\psi_e}, x(i) \rangle \\ &= \langle p(i) - (A_{\psi_e} p(j) + t_{\psi_e}), x(i) \rangle, \end{aligned}$$

by  $t_{\psi_e} = B z_{\psi_e} = (\sum_i s_i B_i) z_{\psi_e}$ . Therefore, as the coordinates of the image of  $p$  and  $s_1, \dots, s_k$  are algebraically independent over  $\mathbb{Q}_{\Gamma_1}$ , we conclude that the  $\Gamma$ -symmetric parallel redrawing polymatroid of  $(G, \psi)$  is obtained from  $\mathbf{LP}(E, \Psi)$  by a Dilworth truncation, given in §3.4.2. By Theorem 3.2, we finally obtain

$$\begin{aligned} &\dim(\text{span}\{P'_{e,\psi}(p) \mid e \in E\}) \\ &= \min\left\{\sum_i (\dim(\text{span}\{U_{e,\psi} \mid e \in E_i\}) - 1) \mid \text{a partition}\{E_1, \dots, E_k\} \text{ of } E\right\} \\ &= \min\left\{\sum_i (h'_\Gamma(E_i) - 1) \mid \text{a partition}\{E_1, \dots, E_k\} \text{ of } E\right\} \\ &= \min\left\{\sum_i h_\Gamma(E_i) \mid \text{a partition}\{E_1, \dots, E_k\} \text{ of } E\right\} \\ &= \hat{h}_\Gamma(E). \end{aligned}$$

□

Summarizing Corollary 10.3 and Theorem 10.4, we complete characterizing the symmetric robustness of drawings with crystallographic symmetry.

**Corollary 10.5.** *Let  $H$  be a  $\Gamma$ -symmetric graph for a symmorphic space group  $\Gamma$  with a generic lattice, and  $(G, \psi)$  be the quotient  $\Gamma$ -gain graph of  $H$ . For almost all  $\Gamma$ -symmetric  $p : V(H) \rightarrow \mathbb{R}^d$ ,  $(H, p)$  is symmetrically robust if and only if the  $\Gamma$ -gain graph obtained from  $G$  by replacing each edge  $e \in E(G)$  by  $d - 1$  parallel copies contains an edge subset  $I$  satisfying the following counting conditions:*

- $|I| = d|V| + k - 1 - \dim(\bigcap_{\gamma \in \Gamma} \ker(A_\gamma - I_d));$

- $|F| \leq d|V(F)| - dc(F) + \dim(\text{span}\{U_{e,\psi_F^\circ} \mid e \in E(G_F^\circ)\}) - 1$  for any nonempty  $F \subseteq I$ ,

where  $(G_F^\circ, \psi_F^\circ)$  is the compressed graph of  $G$  by  $F$  and  $k+2$  denotes the degree of freedom of the lattice.

*Remark 10.1.* As we have remarked,  $\dim(\text{span}\{U_{e,\psi_F^\circ} \mid e \in E(G_F^\circ)\})$  can be deterministically computed in polynomial time. Thus, Corollary 10.5 gives a good characterization of the symmetric robustness. Checking the counting condition can be reduced to a mathematical programming given in (2), which can be deterministically done in polynomial time (see e.g. [29, Theorem 48.4]).  $\square$

### 10.3 Symmetric rigidity with space group symmetry

Let  $C_{\pi/2}$  be the  $2 \times 2$ -matrix representing the 4-fold rotation about the origin over  $\mathbb{R}$ . In § 6.4, we have seen that the idea of characterizing robust drawigns with point group symmetry can be directly applied to characterizing the symmetric infinitesimal rigidity of symmetric 2-dimensional frameworks, if the underlying point group commutes with  $C_{\pi/2}$ . Here, we show an analogous fact in space groups.

The space group  $\Gamma$  we can cope with here is the case when the linear part  $\Gamma_1$  is a group of rotations about the origin. More specifically,  $\Gamma$  falls into five crystallographic group types, called **p1**, **p2**, **p3**, **p4**, **p6** in terms of *Crystallographic notation*. In the subsequent discussion,  $\Gamma$  is assumed to be one of **p2**, **p3**, **p4**, **p6**. (The case of **p1** can be solved in the same manner, but we omit this easier case.)

Let  $(H, p)$  be a  $\Gamma$ -symmetric framework with a  $\Gamma$ -symmetric graph  $H$  and a  $\Gamma$ -symmetric point-configuration  $p$ . An *infinitesimal motion* of  $(H, p)$  is defined as  $m : V(H) \rightarrow \mathbb{R}^2$  satisfying

$$\langle m(i) - m(j), p(i) - p(j) \rangle = 0 \quad \{i, j\} \in E(H). \quad (85)$$

As in the case of the parallel redrawing problem, we are interested in  $\Gamma$ -*symmetric motions*. Here, we say that an infinitesimal motion  $m$  is  $\Gamma$ -*symmetric* if there is  $M \in \overline{\text{Lat}}(\Gamma)$  such that

$$m(\gamma v) = A_\gamma m(v) + M z_\gamma \quad \forall v \in V(H), \forall \gamma \in \Gamma. \quad (86)$$

It can be observed that the infinitesimal rotation  $m_r : V(H) \rightarrow \mathbb{R}^2$  defined by  $m_r(v) = C_{\pi/2} p(v)$  is always a  $\Gamma$ -symmetric infinitesimal motion of  $(H, p)$ . To see this, let  $M = C_{\pi/2} B_\Gamma$ . Then, since  $C_{\pi/2}$  commutes with  $A_\gamma = B_\Gamma K_\gamma B_\Gamma^{-1}$  for any  $\gamma \in \Gamma$ , we have  $C_{\pi/2} B_\Gamma K_\gamma B_\Gamma^{-1} C_{\pi/2}^{-1} = B_\Gamma K_\gamma B_\Gamma^{-1}$ , implying that  $M = C_{\pi/2} B_\Gamma \in \overline{\text{Lat}}(\Gamma)$ . Moreover, for any  $\gamma \in \Gamma$  and  $v \in V(H)$ , we have

$$m_r(\gamma v) = C_{\pi/2} p(\gamma v) = C_{\pi/2} (A_\gamma p(v) + B_\Gamma z_\gamma) = A_\gamma m_r(v) + M z_\gamma,$$

which implies that  $m_r$  satisfies (86) and is indeed a  $\Gamma$ -symmetric motion.

$(H, p)$  is called *infinitesimally rigid* if every possible  $\Gamma$ -symmetric infinitesimal motion is a scalar multiple of  $m_r$ . (This definition is applicable only to  $\Gamma \in \{\mathbf{p2}, \mathbf{p3}, \mathbf{p4}, \mathbf{p6}\}$ .)

As usual, taking a representative vertex  $v$  from each vertex orbit  $\Gamma v$ , (85) is reduced to the system,

$$\langle m(i) - m(\psi_e j), p(i) - p(\psi_e j) \rangle = 0 \quad (87)$$

over all edge orbits from  $\Gamma i$  to  $\Gamma j$  with the gain  $\psi_e$ . Thus, it suffices to consider the following problem on  $\Gamma$ -gain graphs: for a  $\Gamma$ -gain graph  $(G = (V, E), \psi)$  with  $p : V \rightarrow \mathbb{R}^2$ , compute the dimension of the linear space of  $(m, M) \in (\mathbb{R}^2)^V \oplus \overline{\text{Lat}}(\Gamma)$  satisfying

$$\langle m(i) - (A_{\psi_e} m(j) + M z_{\psi_e}), p(i) - (A_{\psi_e} p(j) + t_{\psi_e}) \rangle = 0 \quad \forall (i, j) \in E. \quad (88)$$

Let  $B_1, \dots, B_k \in \mathbb{R}^{d \times d}$  be a basis of  $\overline{\text{Lat}}(\Gamma)$ . As in (79), we shall define a bilinear function  $b_i : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  by  $b_i(\alpha, t) = \langle \alpha, B_i B_\Gamma^{-1} t \rangle$  for  $(\alpha, t) \in \mathbb{R}^d \times \mathbb{R}^d$ , and define  $b : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^k$  by  $b = (b_1, \dots, b_k)^\top$ .

To analyze the system (88), we shall associate a 1-dimensional linear space with each  $e = (i, j) \in E$  as follows:

$$R'_{e, \psi}(p) = U_{e, \psi} \cap \{x \in (\mathbb{R}^2)^V \oplus \mathbb{R}^k \mid \langle C_{\pi/2}(p(i) - (A_{\psi_e} p(j) + t_{\psi_e})), x(i) \rangle = 0\}, \quad (89)$$

where  $U_{e, \psi}$  is as defined in (81)(82). Then, applying the same proof as that of Lemma 10.2, it is easy to check the following.

**Lemma 10.6.** *Let  $\Gamma$  be a 2-dimensional space group whose linear part  $\Gamma_1$  is a group of rotations,  $(G = (V, E), \psi)$  a  $\Gamma$ -gain graph, and  $p : V \rightarrow \mathbb{R}^2$ . Then, the space of  $(m, M) \in (\mathbb{R}^d)^V \oplus \overline{\text{Lat}}(\Gamma)$  satisfying (88) is equal to*

$$2|V| + k - \dim(\text{span}\{R'_{e, \psi}(p) \mid e \in E\}).$$

where  $k = \dim(\overline{\text{Lat}}(\Gamma))$ .

**Theorem 10.7.** *Let  $\Gamma$  be a 2-dimensional space group whose point group  $\Gamma_1$  is a group of rotations and which has the generic lattice  $B_\Gamma$ , and  $(G = (V, E), \psi)$  a  $\Gamma$ -gain graph. Then, for almost all  $p : V \rightarrow \mathbb{R}^2$ ,  $\dim(\text{span}\{R'_{e, \psi}(p) \mid e \in E\}) = \hat{h}_\Gamma(E)$ , where  $h_\Gamma$  is*

$$h_\Gamma(F) = 2|V(F)| - 2c(F) + \dim(\text{span}\{U_{e, \psi_F^\circ} \mid e \in E(G_F^\circ)\}) - 1 \quad (F \subseteq E).$$

*Proof.* Recall that  $C_{\pi/2} B_\Gamma \in \overline{\text{Lat}}(\Gamma)$ . Hence, there is  $s = (s_1, \dots, s_k)^\top \in \mathbb{R}^k$  such that  $\sum_i s_i B_i = C_{\pi/2} B_\Gamma$ . Since the lattice is generic, we may assume that  $\{s_1, \dots, s_k\}$  is algebraically independent over  $\mathbb{Q}_{\Gamma_1}$ .

Let us take any  $p : V \rightarrow \mathbb{R}^2$  such that the coordinates of the image of  $p$  and  $s_1, \dots, s_k$  are algebraically independent over  $\mathbb{Q}_{\Gamma_1}$ . We define a hyperplane  $H'$  of  $(\mathbb{R}^2)^V \oplus \mathbb{R}^k$  by

$$H' = \{x \in (\mathbb{R}^2)^V \oplus \mathbb{R}^k \mid \sum_{v \in V} \langle C_{\pi/2} p(v), x(v) \rangle + \langle s, x(*) \rangle = 0\}.$$

Then, observe that  $R'_{e, \psi}(p) = U_{e, \psi} \cap H'$ . Indeed, for each  $e = (i, j)$  and each  $x \in U_{e, \psi}$ , we have

$$\begin{aligned} & \sum_{v \in V} \langle C_{\pi/2} p(v), x(v) \rangle + \langle s, x(*) \rangle \\ &= \langle C_{\pi/2} p(i), x(i) \rangle + \langle C_{\pi/2} p(j), x(j) \rangle + \langle s, x(*) \rangle \\ &= \langle C_{\pi/2} p(i), x(i) \rangle + \langle C_{\pi/2} p(j), -A_{\psi_e}^{-1} x(i) \rangle + \sum_{\ell} b_\ell(x(i), t_{\psi_e}) s_\ell \\ &= \langle C_{\pi/2}(p(i) - A_{\psi_e} p(j)), x(i) \rangle + \sum_{\ell} \langle B_\ell z_{\psi_e}, x(i) \rangle s_\ell \\ &= \langle C_{\pi/2}(p(i) - A_{\psi_e} p(j)), x(i) \rangle + \langle C_{\pi/2} B_\Gamma z_{\psi_e}, x(i) \rangle \\ &= \langle C_{\pi/2}(p(i) - A_{\psi_e} p(j) - t_{\psi_e}), x(i) \rangle, \end{aligned}$$

where the last two equations follow from  $t_{\psi_e} = B_\Gamma z_{\psi_e}$  and  $C_{\pi/2} B_\Gamma = \sum_\ell s_\ell B_\ell$ . This means that, for any  $x \in U_{e,\psi}$ ,  $x \in H'$  if and only if  $\langle C_{\pi/2}(p(i) - A_{\psi,e}p(j) - t_{\psi_e}), x(i) \rangle = 0$ , implying  $U_{e,\psi} \cap H' = R'_{e,\psi}(p)$ .

By Theorem 9.2,  $\dim(\text{span}\{U_{e,\psi} \mid e \in F\}) = (h_\Gamma + 1)(F)$  for any  $F \subseteq E$ . Thus, by Theorem 3.2, we obtain  $\dim(\text{span}\{R'_{e,\psi}(p) \mid e \in F\}) = \hat{h}_\Gamma(F)$ .  $\square$

Since the space of trivial infinitesimal motions has dimension 1, Lemma 10.6 and Theorem 10.7 imply the following.

**Corollary 10.8.** *Let  $\Gamma$  be a 2-dimensional space group whose linear part  $\Gamma_1$  is a group of rotations and whose lattice is generic. Let  $H$  be a  $\Gamma$ -symmetric graph. Then, for almost all  $\Gamma$ -symmetric  $p : V(H) \rightarrow \mathbb{R}^2$ ,  $(H, p)$  is infinitesimally rigid if and only if the quotient  $\Gamma$ -gain graph  $(G, \psi)$  contains an edge subset  $I$  satisfying the following counting conditions:*

- $|I| = 2|V| + k - 1$ ;
- $|F| \leq 2|V(F)| - 2c(F) + \dim(\text{span}\{R'_{e,\psi_F^\circ} \mid e \in E(G_F^\circ)\}) - 1$  for any nonempty  $F \subseteq I$ ,

where  $k + 2$  denotes the degree of freedom of the lattice.

For  $\Gamma = p2, p3, p4, p6$ ,  $k = 4, 2, 2, 2$ , respectively.

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## References

- [1] N. Biggs. *Algebraic Graph Theory*. Cambridge University Press, 2 edition, 1994.
- [2] C. Borcea and I. Streinu. Periodic frameworks and flexibility. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 466(2121):2633–2649, 2010.
- [3] C. Borcea and I. Streinu. Minimally rigid periodic graphs. *Bulletin of the London Mathematical Society*, 43(6):1093–1103, 2011.
- [4] C. Borcea, I. Streinu, and S. Tanigawa. Periodic body-and-bar frameworks. In *Proc. 24th ACM symposium on Computational Geometry (SoCG2012)*, pages 347–356, 2012.
- [5] T. Brylawski. Constructions. In N. White, editor, *Theory of Matroids*, chapter 7. Cambridge University Press, 1986.
- [6] R. Connelly, P. Fowler, S. Guest, B. Schulze, and W. Whiteley. When is a symmetric pin-jointed framework isostatic? *International Journal of Solids and Structures*, 46(3-4):762–773, 2009.

- [7] T. Dowling. A class of geometric lattices based on finite groups. *Journal of Combinatorial Theory, Series B*, 14(1):61–86, 1973.
- [8] J. Edmonds. Matroid partition. In *Mathematics of the Decision Sciences Part 1*, pages 335–345. AMS, 1968.
- [9] J. Edmonds. Submodular functions, matroids, and certain polyhedra. In R. Guy, H. Hanani, N. Sauer, and J. Schönheim, editors, *Combinatorial Structures and Their Applications*, pages 69–87, 1970.
- [10] P. Fowler and S. Guest. A symmetry extension of maxwell’s rule for rigidity of frames. *International Journal of Solids and Structures*, 37(12):1793–1804, 2000.
- [11] A. Frank. *Connections in Combinatorial Optimization*. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, 2011.
- [12] S. Fujishige. *Submodular Functions and Optimization*. Annals of Discrete Mathematics. Elsevier, 2nd edition, 2005.
- [13] J. L. Gross and T. W. Tucker. *Topological Graph Theory*. Dover, 1987.
- [14] W. V. D. Hodge and D. Pedoe. *Methods of Algebraic Geometry*, volume 1. Cambridge University Press, reissue edition, 3 1994.
- [15] B. Jackson and T. Jordán. Globally rigid circuits of the direction–length rigidity matroid. *Journal of Combinatorial Theory, Series B*, 100(1):1–22, 2010.
- [16] T. Jordán, V. E. Kaszanitzky, and S. Tanigawa. Gain-sparsity and symmetric rigidity in the plane. manuscript.
- [17] J. Kahn and J. Kung. Varieties of combinatorial geometries. *Tran. Amer. Math. Soc.*, 271(2):485–499, 1982.
- [18] L. Lovász. Flats in matroids and geometric graphs. In *Combinatorial surveys: proceedings of the Sixth British Combinatorial Conference*, pages 45–86. Academic Press, 1977.
- [19] L. Lovász and Y. Yemini. On generic rigidity in the plane. *SIAM Journal on Algebraic and Discrete Methods*, 3:91–98, 1982.
- [20] J. Malestein and L. Theran. Generic combinatorial rigidity of periodic frameworks. Technical report, arXiv:1008.1837, 2010.
- [21] J. Malestein and L. Theran. Generic rigidity of frameworks with orientation-preserving crystallographic symmetry. *arXiv:1108.2518*, 2011.
- [22] J. H. Mason. Matroids as the study of geometrical configurations. In M. Aigner, editor, *Higher Combinatorics (Proceedings NATO Advanced Study Institute, 1976)*, pages 133–176. D. Reidel, 1977.

- [23] J. H. Mason. Glueing matroids together: a study of dilworth truncations and matroid analogues of exterior and symmetric powers. In L. Lovász and V. T. Sós, editors, *Algebraic Methods in Graph Theory Vol. II (Colloquium Szeged, 1978)*, pages 519–561. North-Holland, 1981.
- [24] J. Owen and S. Power. Frameworks, symmetry and rigidity. *arXiv:0812.3785*, 2008.
- [25] J. Oxley. *Matroid theory*,. Oxford University Press, USA, 2nd edition, 2011.
- [26] E. Ross. *Geometric and combinatorial rigidity of periodic frameworks as graphs on the torus*. PhD thesis, York University, Toronto, May 2011.
- [27] E. Ross. The rigidity of periodic body-bar frameworks on the three-dimensional fixed torus. Technical report, arXiv:1203.6611, 2012.
- [28] E. Ross, B. Schulze, and W. Whiteley. Finite motions from periodic frameworks with added symmetry. *International Journal of Solids and Structures*, 11–12:1711–1729, 2011.
- [29] A. Schrijver. *Combinatorial optimization: polyhedra and efficiency*. Springer, 2003.
- [30] B. Schulze. *Combinatorial and geometric rigidity with symmetric constraints*. Ph. thesis, York University, 2009.
- [31] B. Schulze. Symmetric versions of laman’s theorem. *Discrete & Computational Geometry*, 44(4):946–972, 2010.
- [32] B. Schulze. Symmetry as a sufficient condition for a finite flex. *SIAM J. Discrete Math.*, 24(4):1291–1312, 2010.
- [33] B. Schulze and W. Whiteley. The orbit rigidity matrix of a symmetric framework. *Discrete Comput. Geom.*, 46(3):561–598, 2011.
- [34] B. Servatius and W. Whiteley. Constraining plane configurations in computer-aided design: combinatorics of directions and lengths. *SIAM Journal on Discrete Mathematics*, 12(1):136–153, 1999.
- [35] S. Tanigawa. Generic rigidity matroids with dilworth truncations. *SIAM J. Discrete Math.*, (To appear).
- [36] T. Tay. Rigidity of multi-graphs. I: Linking rigid bodies in  $n$ -space. *Journal of Combinatorial Theory. Series B*, 36(1):95–112, 1984.
- [37] W. Whiteley. The union of matroids and the rigidity of frameworks. *SIAM Journal on Discrete Mathematics*, 1(2):237–255, 1988.
- [38] W. Whiteley. A matroid on hypergraphs with applications in scene analysis and geometry. *SIAM Journal on Discrete Mathematics*, 4:75–95, 1989.
- [39] W. Whiteley. Some matroids from discrete applied geometry. *Contemporary Mathematics*, 197:171–312, 1996.



- [40] W. Whiteley. Rigidity and scene analysis. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 60, pages 1327–1354. CRC Press, 2 edition, 2004.
- [41] G. Whittle. A generalisation of the matroid lift construction. *Transactions of the American Mathematical Society*, 316(1):141–159, 1989.
- [42] T. Zaslavsky. Biased graphs "I". bias, balance, and gains. *Journal of Combinatorial Theory, Series B*, 47(1):32–52, 1989.
- [43] T. Zaslavsky. Biased graphs "II". the three matroids. *Journal of Combinatorial Theory, Series B*, 51(1):46–72, 1991.
- [44] T. Zaslavsky. Biased graphs "IV": Geometrical realizations. *Journal of Combinatorial Theory, Series B*, 89(2):231–297, 2003.